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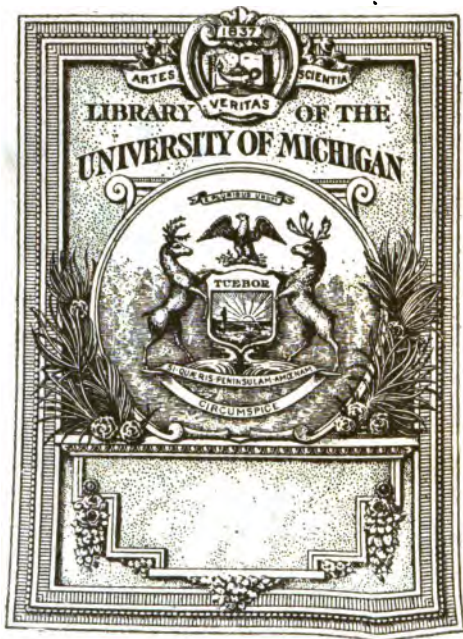
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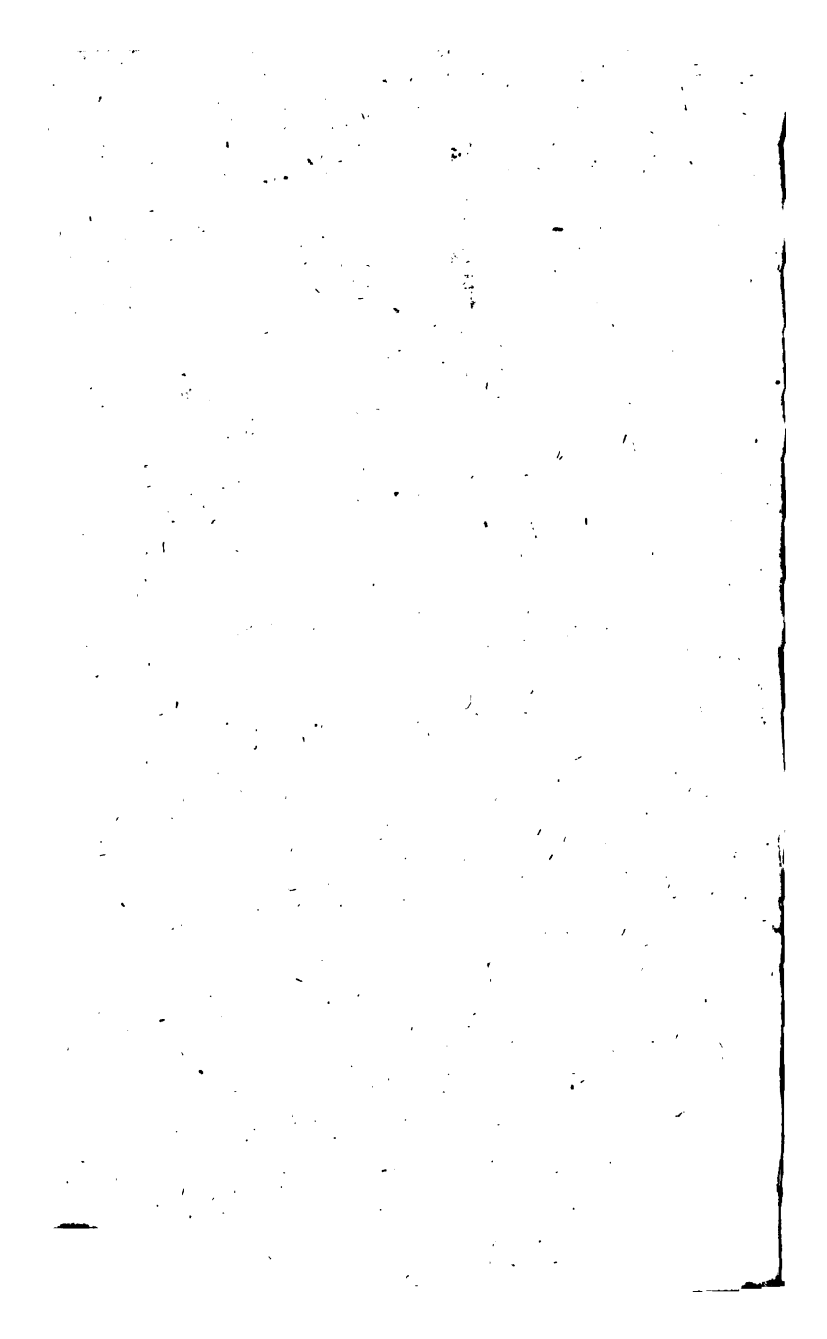
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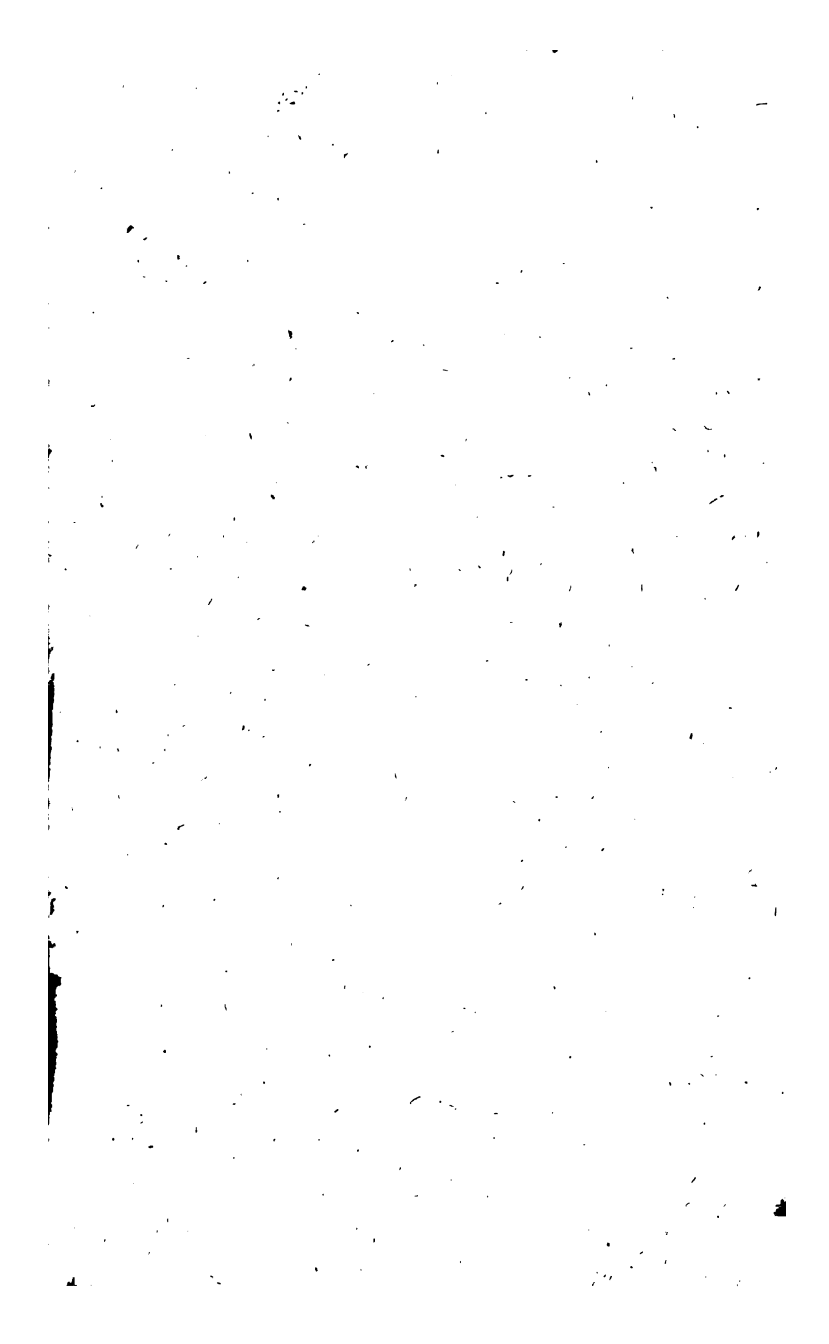


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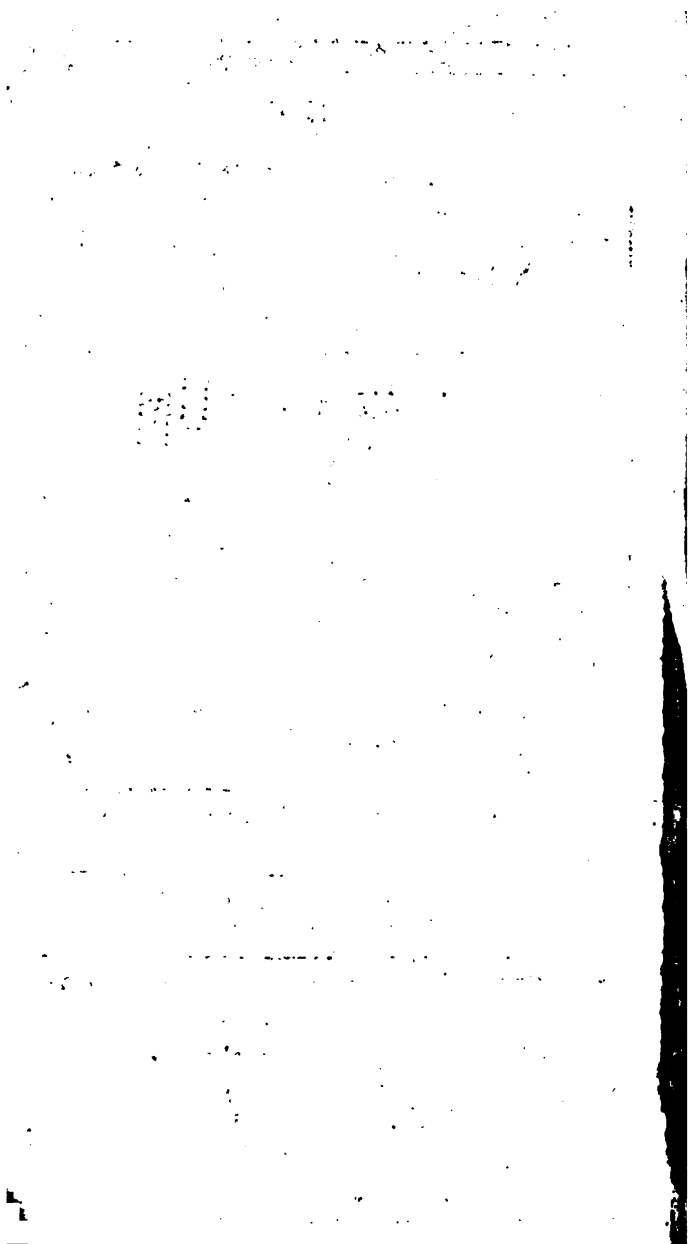
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By Doctor John Keill, F. R. S. And late Professor
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The Third Edition, Translated by Samuel Cunn.
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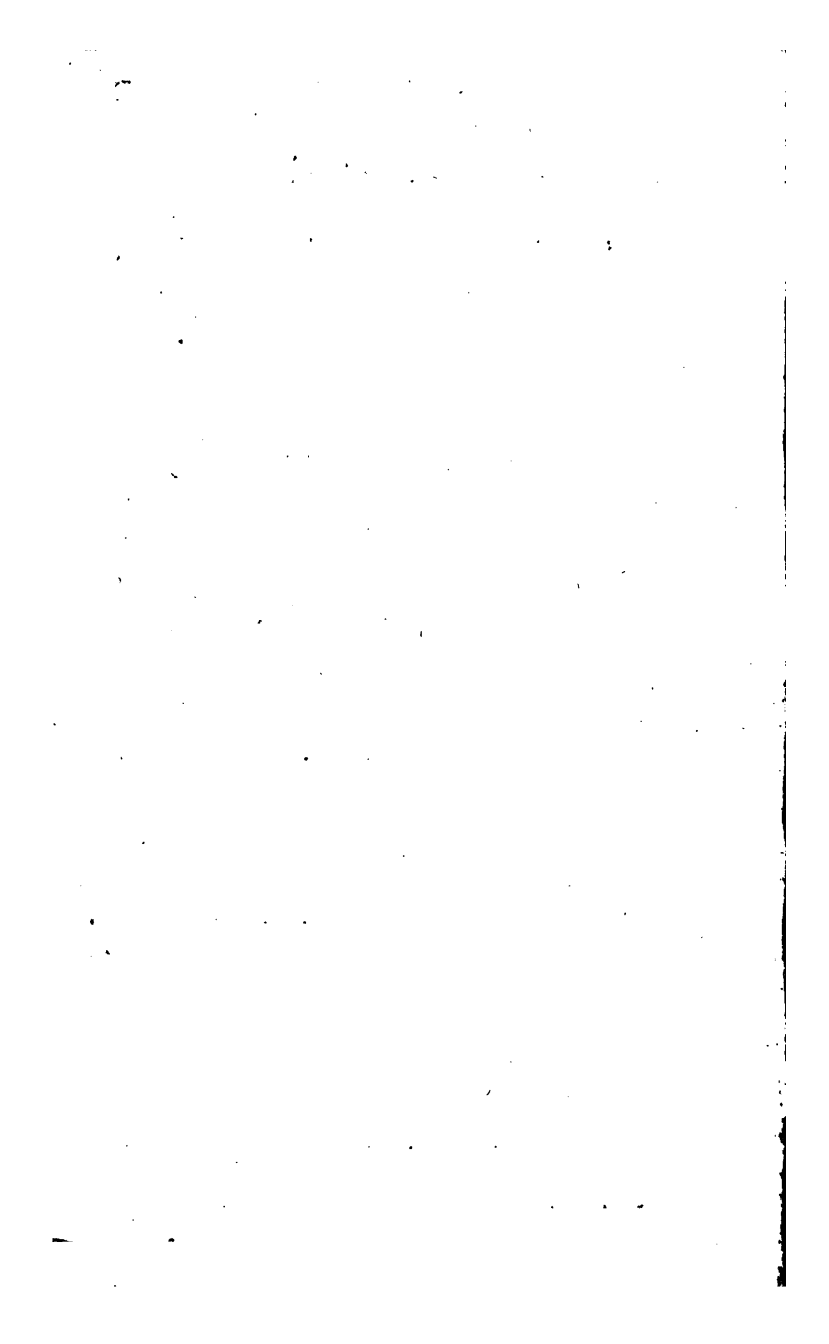
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T H E
E L E M E N T S.
Of Plain and Spherical
T R I G O N O M E T R Y.

D E F I N I T I O N S.



THE Business of Trigonometry is to find the Angles, when the Sides are given, and the Sides, or the Ratio's of the Sides, when the Angles are given, and to find Sides and Angles, when Sides and Angles are given: In order to which, it is necessary that not only the Peripheries of Circles, but also certain Right Lines in and about Circles be supposed divided into some determined Number of Parts.

And so the Ancient Mathematicians thought fit to divide the Periphery of a Circle into 360 Parts (which they call Degrees;) and every

The ELEMENTS of

Degree into 60 Minutes, and every Minute into 60 Seconds: And again, every Second into 60 Thirds, and so on. And every Angle is said to be of such a Number of Degrees and Minutes, as there are in the Arc measuring that Angle.

There are some that would have a Degree divided into centesimal Parts, rather than sexagesimal ones: And it would perhaps be more useful to divide, not only a Degree, but even the whole Circle in a decuple Ratio; which Division may some time or other gain Place. Now, if a Circle contain 360 Degrees, a Quadrant thereof, which is the Measure of a Right Angle, will be 90 of those Parts: And if it contains 100 Parts, a Quadrant will be 25 of these Parts.

The Complement of an Arc is the Difference thereof from a Quadrant.

A Chord, or Subtense, is a Right Line drawn from one End of the Arc to the other.

The Right Sine of any Arc, which also is commonly called only a Sine, is a Perpendicular falling from one End of an Arc, to the Radius down thro' the other End of the said Arc. And is therefore the Semisubtense of double the Arc, viz. $DE = \frac{1}{2} DO$, and the Arc DO is double of the Arc DB . Hence, the Sine of an Arc of 30 Degrees, is equal to the one half of the Radius. For (by 15. El. 4.) the Side of an Hexagon inscribed in a Circle, that is, the Subtense of 60 Degrees is equal to the Radius. A Sine divides the Radius into two Segments CE , EB ; one of which, CE , which is intercepted between the Center and the Right Sine, is the Sine Complement of the Arc DB to a Quadrant, (for CE

PLAIN TRIGONOMETRY.

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$CE = FD$ which is the Sine of the Arc DH , and is called the Cosine. The other Segment BE , which is intercepted between the Right Sine and the Periphery, is called a Versed Sine, and sometimes a Sagitta.

And if the Right Line CG be produced from the Center C , thro' one End D of the Arc, until it meets the Right Line BG , which is perpendicular to the Diameter drawn thro' the other End B of the Arc, then CG is called the Secant, and BG the Tangent of the Arc DB .

The Cosecant and Cotangent of an Arc is the Secant or Tangent of that Arc, which is the Complement of the former Arc to a Quadrant. Note, As the Chord of an Arc, and of it's Complement to a Circle, is the same; so likewise is the Sine, Tangent, and Secant of an Arc the same as the Sine, Tangent, and Secant of the Complement to a Semicircle.

The Sinus Totus is the greatest Sine, or the Sine of 90 Degrees, which is equal to the Radius of the Circle.

A Trigonometrical Canon is a Table, which, beginning from one Minute, orderly expresses the Lengths that every Sine, Tangent, and Secant, have in respect of the Radius, which is supposed Unity; and is conceived to be divided into 10,000,000, or more decimal Parts. And so the Sine, Tangent, or Secant of any Arc, may be had by help of this Table; and contrariwise, a Sine, Tangent, or Secant, being given, we may find the Arc it expresses. Take Notice, That in the following Tract, R signifies the Radius, S a Sine, $Co.$ a Cosine, T a Tangent, and $Cot.$ a Cotangent.

The



The CONSTRUCTION of the Trigonometrical Canon.

PROPOSITION I.

THEOREM.

The two Sides of any Right-angled Triangle being given, the other Side is also given.



OR (by 47 of the first *Element*) $ACq = ABq + BCq$ and $ACq - BCq = ABq$. and interchangeably $ACq - ABq = BCq$. Whence, by the Extraction of the square Root, there is given $AC = \sqrt{ABq + BCq}$ and $AB = \sqrt{ACq - BCq}$. And $BC = \sqrt{ACq - ABq}$.

PROPOSITION II.

PROBLEM.

The Sine DE of the Arc DB, and the Radius CD, being given, to find the Cosine DF.

THE Radius CD and the Sine DE, being given in the Right-angled Triangle CDE, there will be given (by the last *Prop.*) $CE = \sqrt{CDq - DEq} = DF$.

PROPO.

PROPOSITION III.

PROBLEM.

The Sine D E of any Arc D B being given, to find D M or B M the Sine of half the Arc.

D E being given, C E (by the last *Prop.*) will be given, and accordingly E B which is the Difference between the Cosine and Radius. Therefore D E, E B, being given in the Right-angled Triangle D B E, there will be given D B, whose half D M is the Sine of the Arc $D L = \frac{1}{2}$ the Arc B D.

PROPOSITION IV.

PROBLEM.

The Sine B M of the Arc B L being given, to find the Sine of double that Arc.

THE Sine B M being given, there will be given (by *Prop.* 2.) the Cosine C M. But the Triangles C B M, D B E, are equiangular, because the Angles at E and M are Right Angles, and the Angle at B common. Wherefore (by 4. 6.) we have $C B : C M :: B D$, or $2 B M : D E$. Whence, since the three first Terms of this Analogy are given, the fourth also, which is the Sine of the Arc D B, will be known.

Coroll. Hence, $C B : 2 C M :: B D : 2 D E$, that is, the Radius is to double the Cosine of

The ELEMENTS of

of one half of the Arc DB , as the Subtense of the Arc DB , is to the Subtense of double that Arc. Also $CB:2CM::(2BM:2DE::)BM:DE::\frac{1}{2}CB:CM$. Wherefore the Sine of any Arc, and the Sine of it's Double being given, the Cosine of the Arc it self is given.

PROPOSITION V.

The Sines of two Arcs BD , FD , being given, to find FI the Sine of the Sum, as likewise EL , the Sine of their Difference.

LET the Radius CD be drawn, and then CO is the Cosine of the Arc FD , which accordingly is given, and draw OP thro' O parallel to DK . Also let OM , GE , be drawn parallel to CB . Then because the Triangles CDK , COP , CHI , FOH , FOM , are equiangular. In the first Place, $CD:DK::CO:OP$, which Consequentially is known. Also we have $CD:CK::FO:FM$, and so likewise this shall be known. But because $EO=EO$, then will $FM=M'G=ON$. And so $OP+FM=FI$ =Sine of the Sum of the Arcs: And $OP-FM$, that is, $OP-ON=EL$ =Sine of the Difference of the Arcs. W. W. D.

Coroll. Because the Differences of the Arcs BE , BD , BF , are equal, the Arc BD shall be an Arithmetical Mean between the Arcs BE , BF .

P R O P O

PLAIN TRIGONOMETRY.

PROPOSITION VI.

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The same Things being supposed, Radius is to double the Cosine of the mean Arc, as the Sine of the Difference, to the Difference of the Sines of the Extremes.

FOR we have $CD : CK :: FO : FM$, whence by doubling the Consequents $CD : 2CK :: FO : 2FM$, or to FG ; which is the Difference of the Sines EL , FI . W. W. D.

Coroll. If the Arc BD be 60 Degrees, the Difference of the Sines FI , EL , shall be equal to FO the Sine of the Distance. For in this Case, CK , is the Sine of 30 Degrees, Double thereof being equal to Radius; and so since $CD = 2CK$, we shall have $FO = FG$. And consequently, if the two Arcs BE , BF , are Equidistant from the Arc of 60 Degrees, the Difference of the Sines shall be equal to the Sine of the Distance FD .

Coroll. 2. Hence, if the Sines of all Arcs be given, distant from one another by a given Interval, from the Beginning of a Quadrant to 60 Degrees, the other Sines may be found by one Addition only. For the Sine of 61 Degrees = Sine of 59 Degrees + Sine of 1 Degree. And the Sine of 62 Degrees = Sine of 58 Degrees + Sine of 2 Degrees. Also the Sine of 63 Degrees = Sine of 57 Degrees + Sine of 3 Degrees; and so on; *Coroll.*

The ELEMENTS *of*

Coroll. 3. If the Sines of all Arcs, from the Beginning of a Quadrant to any Part of the Quadrant, distant from each other, by a given Interval be given, thence we may find the Sines of all Arcs to the Double of that Part. For Example, Let all the Sines to 15 Degrees be given; then, by the preceding Analogy, all the Sines to 30 Degrees, may be found. For Radius is to double the Cosine of 15 Degrees, as the Sine of 1 Degree, is to the Difference of the Sines of 14 Degrees, and 16 Degrees; so also is the Sine of 3 Degrees, to the Difference between the Sines of 12 and 18 Degrees; and so on continually until you come to the Sine of 30 Degrees.

After the same Manner, as Radius is to double the Cosine of 30 Degrees, or to double the Sine of 60 Degrees, so is the Sine of 1 Degree to the Difference of the Sines of 29 and 31 Degrees :: Sine 2 Degrees, to the Difference of the Sines of 28 and 32 Degrees :: Sine 3 Degrees, to the Difference of the Sines of 27 and 33 Degrees. But in this Case, Radius is to double the Cosine of 30 Degrees, as 1 to $\sqrt{3}$. And accordingly, if the Sines of the Distances from the Arc of 30 Degrees, be multiply'd by $\sqrt{3}$, the Differences of the Sines will be had.

So likewise may the Sines of the Minutes in the Beginning of the Quadrant be found, by having the Sines and Cosines of one and two Minutes given. For as the Radius is
to

to double the Cosine of $2'$:: Sine $1'$: Difference of the Sines of $1'$ and $3'$:: Sine $2'$: Difference of the Sines of $0'$ and $4'$, that is, to the Sine of $4'$. And so the Sines of the four first Minutes being given ; we may thereby find the Sines of the others to $8'$, and from thence to $16'$, and so on.

PROPOSITION VII.

THEOREM.

In small Arcs, the Sines and Tangents of the same Arcs are nearly to one another, in a Ratio of Equality.

FOR because the Triangles CED, CBG, are equiangular, $CE : CB :: ED : BG$. But as the Point E approaches B, EB will vanish in Respect of the Arc BD : Whence CE will become nearly equal to CB. And so ED will be also nearly equal to BG. If

I

EB be less than the $\frac{1}{10,000,000}$ Part of the Radius, then the Difference between the Sine and the Tangent will be also less than the

$\frac{1}{10,000,000}$ Part of the Tangent.

Coroll. Since any Arc is less than the Tangent, and greater than it's Sine, and the Sine and Tangent of a very small Arc, are nearly equal ; it follows that the Arc shall be nearly

The ELEMENTS of

ly equal to it's Sine ; and so in very small Arcs it shall be, as Arc is to Arc, so is Sine to Sine.

PROPOSITION VIII.

To find the Sine of the Arc of one Minute.

THE Side of a Hexagon inscribed in a Circle, that is, the Subtense of 60 Degrees, is equal to the Radius, (*by 15th of the 4th,*) and so the half of the Radius shall be the Sine of the Arc 30 Degrees. Wherefore the Sine of the Arc of 30 Degrees being given, the Sine of the Arc of 15 Degrees may be found, (*by Prop. 3.*) Also the Sine of the Arc of 15 Degrees being given, (*by the same Prop.*) we may have the Sine of 7 Degrees 30 Minutes : So likewise can we find the Sine of the half of this, *viz.* 3 Degrees 45' ; and so on, until twelve Bisections being made, we come to an Arc of $52^{\circ}, 44', 3'', 45''$, whose Cosine is nearly equal to the Radius, in which Case (*as is manifest from Prop. 7.*) Arcs are proportional to their Sines : And so, as the Arc of $52^{\circ}, 44', 3'', 45''$, is to an Arc of one Minute, so shall the Sine before found, be to the Sine of an Arc of one Minute, which therefore will be given. And when the Sine of one Minute is found, then (*by Prop. 2. and 4.*) the Sine and Cosine of two Minutes will be had.

PROPO.

PROPOSITION IX.

THEOREM.

If the Angle BAC, being in the Periphery of a Circle, be bisected by the Right Line AD, and if AC be produced until DE = AD meets it in E: then shall CE = AB.

IN the quadrilateral Figure $ABDC$ (by 22. 3.) the Angles B and ACD are equal to two Right Angles $= DCE + DCA$ (by 13. 1.) Whence the Angle $B = DCE$. But likewise the Angle $E = DAC$ (by 5. i.) $= DAB$ and $DC = DB$. Wherefore the Triangles BAD and CED are congruous, and CE is equal to AB . $W. W. D.$

PROPOSITION X.

THEOREM.

Let the Arcs $AB, BC, CD, DE, EF, \&c.$ be equal; and let the Subtenses of the Arcs $AB, AC, AD, AE, \&c.$ be drawn; then will $AB : AC :: AC : AB + AD :: AD : AC + AE :: AE : AD + AF :: AF : AE + AG.$

LET AD be produced to H , AE to I , AF to K , and AG to L , that the Triangles ACH, ADI, AEK, AFL , be Isosceles ones; then because the Angle BAD is bisected,

bisected, we shall have $DH = AB$, (*by the last Prop.*) so likewise shall $EI = AC$, $FK = AD$, also $GL = AE$.

But the Isosceles Triangles ABC , CHA , DAI , EAK , FAL , because of the equal Angles at the Bases, are equiangular. Wherefore it shall be as $AB : AC :: AC : AH = AB + AD :: AD : AI = AC + AE :: AE : AK = AD + AF :: AF : AL = AE + AG$. W. W. D.

Coroll. 1. Because AB is to AC , as Radius is to double the Cosine of $\frac{1}{2}$ the Arc AB , it shall also be (*by Corol. Prop. 4.*) as Radius is to double the Cosine of $\frac{1}{2}$ the Arc AB , so is $\frac{1}{2}AB : \frac{1}{2}AC :: \frac{1}{2}AC : \frac{1}{2}AB + \frac{1}{2}AD :: \frac{1}{2}AD : \frac{1}{2}AC + \frac{1}{2}AE :: \frac{1}{2}AE : \frac{1}{2}AD + \frac{1}{2}AF$, &c. Now let each of the Arcs AB , BC , CD , &c. be $2'$ then will $\frac{1}{2}AB$ be the Sine of one Minute, $\frac{1}{2}AC$ the Sine of $2'$ Minutes, $\frac{1}{2}AD$ the Sine of $3'$ Minutes; $\frac{1}{2}AE$ the Sine of $4'$, &c. Whence if the Sines of one and two Minutes be given, we may easily find all the other Sines in the following Manner.

Let the Cosine of the Arc of one Minute, that is, the Sine of the Arc of 89 Deg. $59'$, be called Q , and make the following Analogies; $R : 2Q :: \text{Sin. } 2' : S. 1' + S. 3'$. Wherefore the Sine of 3 Minutes will be given. Also $R : 2Q :: S. 3' : S. 2' + S. 4'$. Wherefore the $S. 4'$ is given; and $R : 2Q :: S. 4' : S. 3' + S. 5'$; and so the Sine of $5'$ will be had.

Likewise

Likewise $R : 2 Q :: S. 5' : S. 4' + S. 6'$; and so we shall have the Sine of $6'$. And in the like Manner, the Sines of every Minute of the Quadrant will be given. And because the Radius, or the first Term of the Analogy is Unity, the Operations will be with great Ease and Expedition calculated by Multiplication, and contracted by Addition. When the Sines are found to 60 Degrees, all the other Sines may be had by Addition only, (by *Cor. 1. Prop. 6.*)

The Sines being given, the Tangents and Secants may be found from the following Analogies, (in the Figure for the Definitions;) because the Triangles CED , CBG , CHI , are equiangular, we have.

$CE:ED::CB:BG$; that is, $Cof.:S::R:T$.
 $GB:BC::CH:HI$; that is, $T:R::R:Cot$.
 $CE:CD::CB:CG$; that is, $Cof.:R::R$:
 Secant.

$DE:CD::CH:CI$; that is, $S:R::R:Cofec$.

SCHOLIUM.

That great Geometrician and incomparable Philosopher, Sir Isaac Newton, was the first that laid down a Series converging, in infinitum; from which, having the Arts given, their Sines may be found. Thus if an Arc be called A, and the Radius be an Unit, the Sine thereof will be found to be.

A^3 A^5 A^7 A^9
 $1 - \frac{A^3}{3} + \frac{A^5}{5} - \frac{A^7}{7} + \frac{A^9}{9} - \dots$
 $1.2.3. \quad 1.2.3.4.5. \quad 1.2.3.4.5.6.7. \quad 1.2.3.4.5.6.7.8.9.$
And

And the Cofine,

$$1 - \frac{A^2}{1.2} + \frac{A^4}{1.2.3.4} - \frac{A^6}{1.2.3.4.5.6} + \frac{A^8}{1.2.3.4.5.6.7.8} - \text{&c.}$$

These Series in the Beginning of the Quadrant when the Arc A is but small, soon converge. For in the Series for the Sine, if A does not exceed 10 Minutes, the two first Terms thereof, viz. $A - \frac{1}{6} A^3$ gives the Sine to 15 Places of Figures. If the Arc A be not greater than one Degree, the three first Terms will exhibit the Sine to 15 Places of Figures; and so the said Series are very useful for finding the first and last Sines of the Quadrant. But the greater the Arc A is, the more are the Terms of the Series required to have the Sine in Numbers true to a given Place of Figures. And then when the Arc is nearly Equal to the Radius, the Series Converges very slow. And therefore, to remedy this, I have devised other Series, similar to the Newtonian one, wherein, I suppose, the Arc, whose Sine is sought, is the Sum and Difference of two Arcs, viz. $A + z$, or $A - z$: And let the Sine of the Arc A, be called a, and the Cofine b. Then the Sine of the Arc $A + z$ will be expressed thus:

$$1. \ a + \frac{bz}{1} - \frac{az^2}{1.2} + \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} + \frac{bz^5}{1.2.3.4.5} - \text{&c.}$$

And the Cofine is,

$$2. \ b - \frac{az}{1} + \frac{bz^2}{1.2} - \frac{az^3}{1.2.3} + \frac{bz^4}{1.2.3.4} - \frac{az^5}{1.2.3.4.5} + \frac{bz^6}{1.2.3.4.5.6} - \text{&c.}$$

In

In like Manner the Sine of the Arc A — z is

$$3. a - \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6} \quad \text{&c.}$$

&c.

And the Cosine is,

$$4. b + \frac{az}{1} - \frac{bz^2}{1.2} + \frac{az^3}{1.2.3} - \frac{bz^4}{1.2.3.4} + \frac{az^5}{1.2.3.4.5} - \frac{bz^6}{1.2.3.4.5.6} \quad \text{&c.}$$

The Arc A is an Arithmetical Mean between the A — z and A + z. And the Difference of the Sines are,

$$5. \frac{bz}{1} - \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} - \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5} - \frac{az^6}{1.2.3.4.5.6} \quad \text{&c.}$$

$$6. \frac{bz}{1} + \frac{az^2}{1.2} - \frac{bz^3}{1.2.3} + \frac{az^4}{1.2.3.4} - \frac{bz^5}{1.2.3.4.5} + \frac{az^6}{1.2.3.4.5.6} \quad \text{&c.}$$

&c.

Whence the Difference of the Differences or second Difference.

$$7. \text{Produce } \frac{2az^2}{1.2} - \frac{2az^4}{1.2.3.4} + \frac{2az^6}{1.2.3.4.5.6} \quad \text{&c.}$$

$$\text{Or, } 2a \times \frac{z^2}{1.2} - \frac{z^4}{1.2.3.4} + \frac{z^6}{1.2.3.4.5.6} \quad \text{&c.}$$

Which Series is equal to double the Sine of the Mean Arc, drawn into the versed Sine of the Arc z, and Converges very soon. So that if z be the first Minute of the Quadrant, the first Term of the Series gives the second Difference to 15 Places of Figures, and the second Term to 25 Places.

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From hence, if the Sines of the Arcs distant one Minute from each other be given. The Sines of all the Arcs, that are in the same Progression may be found by an exceeding easy Operation.

In the first and second Series, if $A = 0$; then shall $a = 0$, and b its Cosine, will become Radius, or 1. And hence, if the Terms wherein a is, are taken away, and 1 be put instead of b , the Series, will become the Newtonian. In the third and fourth Series, if A be 90 Degrees, we shall have $b = 0$, and $a = 1$. Whence again, taking away all the Terms wherein b is, and putting 1 instead of a , we shall have the Newtonian Series arise.

Note, All the said Series easily flow from the Newtonian ones. By the fifth Proposition.

PROPOSITION XI.

In a Right-angled Triangle, if the Hypotenuse be made the Radius, then are the Sides the Sines of their opposite Angles; and if either of the Legs be made the Radius, then the other Leg is the Tangent of it's opposite Angle, and the Hypotenuse is the Secant of that Angle.

IT is manifest that CB is the Sine of the Arc CD , and AB the Cosine thereof; but the Arc CD is the Measure of the Angle A and the Complement of the Measure of the Angle C . Moreover, if AB , in the Figure to this Proposition, be supposed Radius, then BC is the Tangent, and AC the Secant of the Arc BD , which is the Measure of the Angle A . So also if BC be made the Radius, then
is

is BA the Tangent, and AC the Secant of the Arc BE , or Angle C . $W. W. D.$ Therefore as AC being taken as some given Measure is to BC taken in the same Measure; so shall the Number 10000000 Parts in which the Radius is supposed to be divided, be to a Number expressing in the same Parts the Length of the Sine of the Angle A ; that is, it will be.

as $AC : BC :: R : S, A.$
by the same Reason, as $AC : BA :: R : S, C.$
also as $AB : BC :: R : T, A.$
and $BC : BA :: R : T, C.$

And so if any three of these Proportionals be given, the fourth may be found by the *Rule of Three*.

PROPOSITION XII.

The Sides of Plain Triangles are as the Sines of their opposite Angles.

IF the Sides of a Triangle inscribed in a Circle be bisected by perpendicular Radii, then shall the half Sides be the Sines of the Angles at the Periphery; for the Angle BDC at the Center is double of the Angle BAC at the Periphery; (by 20 *El. lib. 3.*) and so the half of every of them, *viz.* $BDE = BAC$, and BE is the Sine thereof. For the same Reason, BF shall be the Sine of the Angle BCA , and AG the Sine of the Angle ABC .

In a Right-angled Triangle we have $BD = \frac{1}{2} BC = \text{Radius}$ (by 31 *Eucl. 3.*) but Radius

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dus is the Sine of a Right Angle: Whence $\frac{1}{2} BC$ is the Sine of the Angle A.

In an Obtuse-angled Triangle, let BL, CL, be drawn, and then the Angle L shall be the Complement of the Angle A to two Right Angles, (by 22 El. 3.) and so they shall both have the same Sine. But the Angle BDE, (whose Sine is BE) = Angle L. Therefore BE shall be the Sine of the Angle BAC. And so in every Triangle, the Halves of the Sides are the Sines of the opposite Angles; but it is manifest that the Sides are to one another as their Halves. W. W. D.

PROPOSITION XIII.

In a plain Triangle, the Sum of the Legs, the Difference of the Legs, the Tangent of the half Sum of the Angles at the Base, and the Tangent of one half their Difference, are proportional.

LET there be a Triangle ABC, whose Legs are AB, BC, and Base AC. Produce AB to H, so that BH = BC, then shall AH be the Sum of the Legs; and if you make BI = BA, then IH will be the Difference of the Legs. Also the Angle HBC = Angles A + ACB, (by 32. El. 1.) and so EBC the half thereof = half the Sum of the Angles A and ACB, and it's Tangent (putting the Radius = EB) is EC. Again, let BD be drawn parallel to AC, and make HF = CD. Then since HB = CB, we shall have (by 4 El. 1.) the Angle HBF = CBD = BCA. (by 29 El. 1.) Also the Angle HBD = Angle

$\angle A$; whence FBD shall be the Difference of the Angles A and ACB ; and EBD , whose Tangent is ED , half their Difference. Let IG be drawn thro' I parallel to AC or BD , and then (by 2 *El.* 6.) $AB : BI :: CD : DG$, but $AB = BI$; whence we shall have $CD = DG$, but $CD = HF$, and so $HF = DG$, and consequently, $HG = DF$, and $\frac{1}{2}HG = \frac{1}{2}DF = DE$; and because the Triangles AHC , IHG , are equiangular, it shall be as $AH : IH :: HC : HG :: \frac{1}{2}HC : \frac{1}{2}HG :: EC : ED$. That is, AH the Sum of the Legs, to IH the Difference of the Legs, shall be as EC the Tangent of one half the Sum of the Angles at the Base, to ED the Tangent of one half their Difference. *W. W. D.*

PROPOSITION XIV.

In a plain Triangle, the Base, the Sum of the Sides, the Difference of the Sides, and the Difference of the Segments of the Base, are proportional.

LET DC be the Base of the Triangle BCD , about the Center B , with the Radius BC , let a Circle be described. Produce DB to G , and from B let fall BE perpendicular to the Base; then shall $DG = DB + BC =$ Sum of the Sides, and $DH =$ Difference of the Sides; and DE, CE , are the Segments of the Base whose Difference is DF ; because (by *Cor. Prop.* 38. *El.* 3.) the Rectangle under DG and DF , is equal to the Rectangle under DG, DH , it shall be

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be (by 16 El. 6.) as $DC : DG :: DH : DF$. W. W. D.

PROBLEM.

The Sum and Difference of any two Quantities being given, to find the Quantities themselves.

IF one half of the Sum be added to one half of the Difference, the Aggregate shall be equal to the greater of the Quantities ; and if from one half of the Sum be taken one half of the Difference, the Residue shall be equal to the lesser of the Quantities. For let there be two Quantities AB , BC , and let there be taken $AD = BC$, then DB will be their Difference, and AC their Sum ; which, bisected in E , gives AE or EC the half Sum, and DE or EB the half Difference. Hence $AB = AE + EB = \text{the half Sum} + \text{the half Difference}$, and $BC = CE - EB = \text{the half Sum} - \text{the half Difference}$.

In any plain Triangle if two Angles be given, the third Angle is also given, because it is their Complement to two Right Angles.

If one of the acute Angles of a Right-angled Triangle be given, the other acute Angle will be given, because it is the Complement of the given Angle to a Right Angle.

And if two Sides of a Right-angled Triangle be given, the other Side may be found by the first Proportion without a Canon.



The

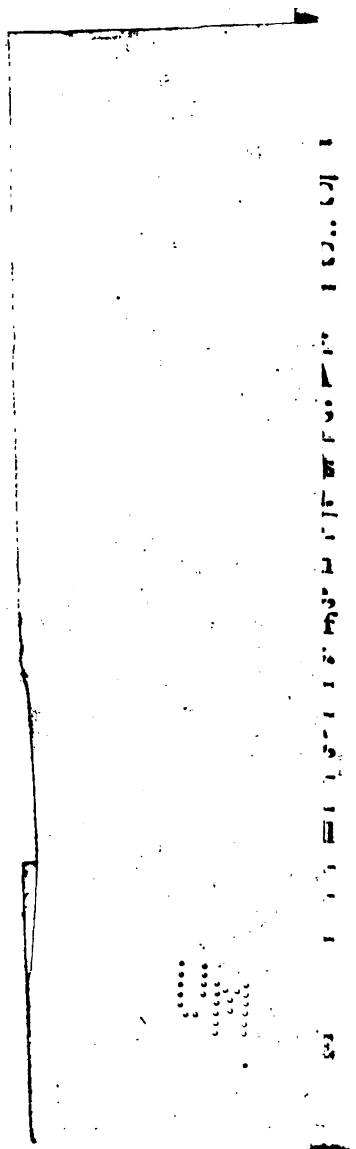
The Trigonometrical Solutions of a Right-angled Triangle, may be as follow. Vid. Fig. A.

	Given	Sought	Make as
1	The Legs A B and B C	The Angles.	$AB : BC :: R : T$ of the Angle A, whose Complement is the Angle C.
2	The Leg A B and the Hypothenuse A C.	The Angles.	$AC : AB :: R : S$, C whose Complement is the Angle A.
3	The Leg A B and the Angle A.	The Other Side B C, and the Hypothenuse A C	$R : T. A :: AB : BC.$ $S, C : R :: AB : AC.$
4	The Hypothenuse A C, and the Angle C.	The Leg A B.	$R : S, C :: AC : AB.$

The

The Trigonometrical Solutions of oblique-angled Triangles. Vid. Fig. to Prop. 12.

	Given	Sought	Make as
1	The Angles A, B, C, and the Side A B.	The Sides B C and A C.	$S, C : S A :: A B : B C$. Also $S, C : S, B :: A B : A C$. But when two Angles are given the third is also given whence the Case wherein two Angles and a Side are given, to find the rest, falls into this Case.
2	All the Angles A, B, C.	All the Sides A B, A C, B C.	$S, C : S, A :: A B : B C$. And $S, C : S, B :: A B : B C$. Whence if the Angles are given, the Proportions of the Sides may be found, but not the Sides themselves, unless one of them be first known.
3	The two Sides A B, B C, and the Angle opposite to one of them.	The Angles A and B.	$A B : B C :: S, C : S, A$; which therefore may be found. When A B the Side opposite to C, the given Angle is longer than B C the Side opposite to the sought Angle, the sought Angle is less than a right one. But when it is shorter, because the Sine of an Angle, and that of its Complement to two Right Angles, is the same, the Species of the Angle A must be first known, or the Solution will be ambiguous.



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Given	Sought	Make as
The two Sides A B, B C, and the inter-jacent Angle B.	The Angles A and C.	<i>Vid. Fig. to Prop. 13.</i> BC $+ AB : BC - AB ::$ $A + C \quad A - C$ $T, \quad \text{---} : T, \quad \text{---}$ $\quad \quad \quad 2 \quad \quad \quad 2$ Whence is known the Difference of the Angles A and C, whose Sum is given; and so (by the <i>Prob. following Prop. 14.</i>) the Angles themselves will be given.
All the Sides A B, B C, A C.	The Angles.	<i>Vid. Fig. B.</i> Let the Perpendicular be drawn from the Vertex to the Base, and find the Segments of the Base by <i>Prop. 14. viz.</i> Make as $BC : AC + AB :: AC - AB : DC - DB$. And so B D, D C, are given from this Analogy; and thence the Angles A B D, A D C, will be given by the Resolution of Right-angled Triangles.

D

T H E



THE
ELEMENTS
OF
Spherical TRIANGLES.

DEFINITIONS.

I.  *THE Poles of a Sphere are two Points in the Superficies of the Sphere that are the Extremes of the Axis.*

II. *The Pole of a Circle in a Sphere, is a Point in the Superficies of the Sphere, from which all Right Lines that are drawn to the Circumference of the Circle, are equal to one another.*

III. *A great Circle in a Sphere, is that whose Plane passes thro' the Center of the Sphere, and whose Center is the same as that of the Sphere.*

IV. *A*

IV. *A spherical Triangle is a Figure comprehended under the Arcs of three great Circles in a Sphere.*

V. *A spherical Angle is that which, in the Superficies of the Sphere, is contained under two Arcs of great Circles; and this Angle is equal to the Inclination of the Planes of the said Circles.*



PROPOSITION I.

Great Circles ACB , AFB , mutually bisect each other.



OR since the Circles have the same Center, their common Section shall be a Diameter of each Circle, and so will cut them into two equal Parts.

Coroll. Hence the Arcs of two great Circles in the Superficies of the Sphere, being less than Semicircles, do not comprehend a Space; for they cannot, unless they meet each other in two opposite Points in a Semicircle.

PROPOSITION II.

If from the Pole C of any Circle AFB , be drawn a Right Line CD to the Center thereof, the said Line will be perpendicular to the Plane of that Circle. Vid. Fig. to Prop. 1.

LET there be drawn any Diameters EF , GH , in the Circle AFB ; then because

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in the Triangles $C D F$, $C D E$, the Sides $C D$, $D F$, are equal to the Sides $C D$, $D E$, and the Base $C F$, equal to the Base $C E$; (*by Def. 2.*) then (*by 4 El. 1.*) shall the Angle $C D F = \text{Angle } C D E$; and so each of them will be a Right Angle. After the same Manner we demonstrate that the Angles $C D G$, $C D H$, are Right Angles; and so (*by 4 El. 11.*) $C D$ shall be perpendicular to the Plane of the Circle $A F E$. W. W. D.

Coroll. 1. A great Circle is distant from it's Pole by the Interval of a Quadrant; for since the Angles CDG , CDF , are Right Angles, the Measures of them, *viz.* the Arcs CG , CF , will be Quadrants.

2. Great Circles that pass thro' the Pole of some other Circle, make Right Angles with it; and contrariwise, if great Circles make Right Angles with some other Circle, they shall pass thro' the Poles of that other Circle, for they must necessarily pass thro' the Right Line DC .

PROPOSITION III.

If a great Circle ECF be described about the Pole A ; then the Arc CF intercepted between AC , AF , is the Measure of the Angle $C A F$, or $C D F$, Vid. Fig. to Prop. 1.

THE Arcs AC , AF , (*by Cor. 1. Prop. 2.*) are Quadrants, and consequently the Angles ADC , ADF , are Right Angles. Wherefore (*by Def. 6. El. 11.*) the Angle $C D F$,

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CDE, (whose Measure is the Arc CF) is equal to the Inclination of the Planes ACB, AFB, and also equal to the spherical Angle CAF, or CBF. W. W. D.

Coroll. 1. If the Arcs AC, AF, are Quadrants, then shall A be the Pole of the Circle passing thro' the Points C and F; for AD, is at Right Angles to the Plane FDC. (by 14 El. II.)

The Vertical Angles are equal, for each of them is equal to the Inclination of the Circles; also the adjoining Angles are equal to two Right Angles.

PROPOSITION IV.

Triangles shall be equal and congruous, if they have two Sides equal to two Sides, and the Angles comprehended by the two Sides also equal.

PROPOSITION V.

Also Triangles shall be equal and congruous, if one Side together with the adjacent Angles in one Triangle, be equal to one Side, and the adjacent Angles of the other Triangle.

PROPOSITION VI.

Triangles mutually Equilateral, are also mutually equiangular.

PROPO:

PROPOSITION VII.

In Isosceles Triangles, the Angles at the Base are equal.

PROPOSITION VIII.

And if the Angles at the Base be equal, then the Triangle shall be Isosceles.

These four last Propositions are demonstrated in the same Manner, as in plain Triangles.

PROPOSITION IX.

Any two Sides of a Triangle are greater than the third.

FOR the Arc of a great Circle, is the shortest Way, between any two Points in the Superficies of the Sphere.

PROPOSITION X.

A Side of a spherical Triangle is less than a Semicircle.

LET AC, AB, the Sides of the Triangles ABC be produced till they meet in D; then shall the Arc ACD, which is greater than the Arc AC, be a Semicircle.

PROPO.

PROPOSITION XL

The three Sides of a spherical Triangle are less than a whole Circle.

FOR $DB + DC$ is greater than BC , (by Prop. 9.) and adding on each Side $BA + AC$, $DBA + DCA$, that is, a whole Circle will be greater than $AB + BC + AC$, which are the three Sides of the spherical Triangle ABC .

PROPOSITION XIK

In any spherical Triangle ABC , the greater Angle A is subtended by the greater Side.

MAKE the Angle $BAD = \text{Angle } B$; then shall $AD = BD$, (by 8 of this;) and so $BDC = DA + DC$, and these Arcs are greater than AC . Wherefore the Side BC , that subtends the Angle BAC , is greater than the Side AC , that subtends the Angle B .

PROPOSITION XIII.

In any spherical Triangle ABC , if the Sum of the Legs AB and BC be greater, equal, or less, than a Semicircle, then the internal Angle at the Base AC shall be greater, equal, or less, than the external and opposite Angle BCD , and so the Sum of the Angles A and ACB shall also be greater, equal, or less, than two Right Angles.

FIRST, let $AB + BC = \text{Semicircle} = AD$, then shall $BC = BD$, and the Angles

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gles BCD and D equal, (by 8 of this,) and therefore the Angle BCD shall be $=$ Angle A .

Secondly, Let $AB + BC$ be greater than ABD ; then shall BC be greater than BD and so the Angle D (that is, the Angle A by 12 of this) shall be greater than the Angle BCD . In like Manner we demonstrate, if $AB + BC$ be together less than a Semicircle, that the Angle A will be less than the Angle BCD . And because the Angles BCI and BCA , are $=$ two Right Angles; if the Angle A be greater than the Angle BCD then shall A and BCA , be greater than two Right Angles; if the Angle $A = BCD$, then shall A and BCA be equal to two Right Angles. And if A be less than BCD , then will A and BCA be less than two Right Angles. W.W.D.

PROPOSITION XIV.

In any spherical Triangle GHD , the Poles of the Sides being joined by great Circles, do constitute another Triangle XMN , which is the Supplement of the Triangle GHD viz. the Sides NX , XM , and NM , shall be Supplements of the Arcs that are the Measures of the Angles D , G , H , to the Semicircles; and the Arcs that are the Measure of the Angles M , X , N , will be the Supplements of the Sides GH , GD , and HD to Semicircles.

FROM the Poles G , H , D , let the great Circles $XCAM$, $TMNO$, $XKBN$ be

be described; then because G is the Pole of the Circle $XCAM$, we shall have $GM =$ Quadrant, (by *Cor. 1 Prop. 2.*) and since H is the Pole of the Circle TMO , then will HM be also a Quadrant; and so (by *Cor. 1. Prop. 3.*) M shall be the Pole of the Circle GH . In like Manner, because D is the Pole of the Circle XBN , and H the Pole of the Circle TMN , the Arcs DN , HN , will be Quadrants; and so (by *Cor. 1. Prop. 3.*) N shall be the Pole of the Circle HD . And because $G X$, $D X$, are Quadrants, X will be the Pole of the Circle GD . These Things premised.

Because $NK =$ Quadrant, (*Cor. 1. Prop. 2.*) then will $NK + XB$, that is, $NX + KB =$ two Quadrants, or a Semicircle; and so NX is the Supplement of the Arc KB , or of the Measure of the Angle $H D G$ to a Semicircle. In like Manner, because $MC =$ Quadrant, and $XA =$ Quadrant, then will $MC + XA$; that is, $XM + AC =$ two Quadrants, or a Semicircle; and consequently XM is the Supplement of the Arc AC , which is the Measure of the Angle $H G D$. Likewise, since MO , NT , are Quadrants, we shall have $MO + NT = OT + NM =$ Semicircle. And therefore NM is the Supplement of the Arc OT , or of the Measure of the Angle $G H D$, to a Semicircle. **W. W. D.**

Moreover, because DK , HT , are Quadrants, $DK + HT$, or $KT + HD$, are equal to two Quadrants, or a Semicircle. Therefore KT , or the Measure of the Angle $X N M$, is the Supplement of the Side HD to a Semicircle.

micircle. After the same Manner it is demonstrated, that OC , the Measure of the Angle XMN , is the Supplement of the Side GH , and BA the Measure of the Angle X , is the Supplement of the Side GD . W. W. D.

PROPOSITION XV.

Equiangular spherical Triangles are also equilateral.

FOR their Supplementals (*by 14 of this*) are equilateral; and therefore equiangular also; and so themselves are likewise equilateral, (*by Part 2. Prop. 14.*)

PROPOSITION XVI.

The three Angles of a spherical Triangle, are greater than two Right Angles, and less than six.

FOR the three Measures of the Angles G, H, D , together with the three Sides of the Triangle XMN , make three Semicircles, (*by 14 of this;*) but the three Sides of the Triangle XMN , are less than two Semicircles, (*by 11 of this.*) Wherefore the three Measures of the Angles G, H, D , are greater than a Semicircle; and so the Angles G, H, D , are greater than two Right Angles.

The second Part of the Proposition is manifest, for in every spherical Triangle, the external and internal Angles together, only make six Right Angles; wherefore the internal Angles are less than six Right Angles.

PROPO

PROPOSITION XVII.

If from the Point R, not being the Pole of the Circle AFBE, there fall the Arcs RA, RB RG, RV, of great Circles to the Circumference of that Circle; then the greatest of those Arcs is RA, which passes thro' the Pole C thereof; and the Remainder of it is the least; and those that are more distant from the greatest are less than those which are nearer to it, and they make an obtuse Angle with the former Circle AFB, on the Side next to the greatest Arc. Vid. Fig. to Prop. I.

BECAUSE C is the Pole of the Circle AFB, then shall CD and RS, which is parallel thereto, be perpendicular to the Plane AFB. And if SA, SG, SV, be drawn, then shall SA (by 7 El. 3.) be greater than SG, and SG greater than SV. Whence in the Right-angled plain Triangles RSA, RSG, RSV, we shall have $RSq + SAq$, or RAq , greater than $RSq + SGq$, or RGq ; and so RA will be greater than RG, and the Arc RA greater than the Arc RG. In like Manner, $RSq + SGq$, or RGq shall be greater than $RS + SVq$, or RVq ; and so RG shall be greater than RV, and the Arc RG greater than the Arc RV.

2dly, The Angle RGA is greater than the Angle CGA, which is a Right Angle (by Cor. Prop. 3.) and the Angle RVA is greater than the Angle CVA, which also is a Right Angle. Therefore the Angles RGA, RVA, are obtuse Angles.

PROPOSITION XVIII.

In a spherical Triangle right-angled at A, the Legs containing the Right Angle, are of the same Affection with the opposite Angles; that is, if the Legs be greater or less than Quadrants, then accordingly will the Angles opposite to them be greater or less than Right Angles. Vid. Fig. to Prop. 1.

FOR if A C be a Quadrant, then will C be the Pole of the Circle A F B, and the Angles A G C, or A V C, will be Right Angles. If the Leg A R be greater than a Quadrant, then shall the Angle A G R be greater than a Right Angle, (*by 17 of this*;) and if the Leg A X be less than a Quadrant, the Angle A G X shall be less than a Right Angle.

PROPOSITION XIX.

If two Legs of a right-angled spherical Triangle be of the same Affection; (and consequently the Angles,) that is, if they are both less or both greater than a Quadrant, then will the Hypothenuse be less than a Quadrant. Vid. Fig. to Prop. 1.

IN the Triangle A R V, or B R V, let F be the Pole of the Leg A R, then will R F be a Quadrant, which is greater than R V, (*by 17 of this*.)

PROP^o.

PROPOSITION XX.

If they be of a different Affection; then shall the Hypothenuſe be greater than a Quadrant.
Vid. Fig. to Prop. 1.

FOR in the Triangle ARG , the Hypothenuſe RG is greater than RF , which is a Quadrant.

PROPOSITION XXI.

If the Hypothenuſe be greater than a Quadrant, then the Legs of the Right Angle, and ſo the Angles oppoſite to them, are of different Affection; but if leſſer, of the ſame Affection.
Vid. Fig. to Prop. 1.

THIS Proposition being the Converſe of the former ones, eaſily follow from them.

PROPOSITION XXII.

In any ſpherical Triangle ABC , if the Angles at the Baſe B and C , be of the ſame Affection, then the Perpendicular falls within the Triangle; and if they be of a different Affection, the Perpendicular falls without the Triangle.

IN the firſt Caſe, if the Perpendicular does not fall within, let it fall without the Triangle, (as in Fig. 2.) then in the Triangle ABP , the Side AP is of the ſame Affection with the Angle B . And in like Manner, in the Triangle ACP , AP , is of the ſame Affection with the Angle C . Therefore
ſince

since ABC , and ACP , are of the same Affection, the Angles ABC , ACB , shall be of different Affection; which is contrary to the Hypothesis.

In the second Case, if the Perpendicular does not fall without, let it fall within, (as in *Fig. 1.*) Then in the Triangle ABP , the Angle B is of the same Affection with the Leg AP . So likewise, in the Triangle ACP , the Angle C is of the same Affection with AP ; and therefore the Angles B and C are of the same Affection; which is contrary to the Hypothesis.

PROPOSITION XXIII.

In spherical Triangles BAC , BHE , right angled at A and H , if the same acute Angle B be at the Base BA , or BH , then the Sine of the Hypotenuse shall be proportional to the Sines of the perpendicular Arcs.

FOR the Right Lines CD , EF , being perpendicular to the same Plane, are parallel. Also FR , DP , perpendicular to the Radius OB , are likewise parallel; wherefore the Planes of the Triangles EFR , CDP are also parallel, (by 15 *El. 11.*) Wherefore CP , ER , the common Sections of those Planes with the Plane passing thro' BE , CO , will be parallel, (by 16 *El. 11.*) Therefore the Triangles CDP , EFR , shall be equiangular. Wherefore CP the Sine of the Hypotenuse BC , is to CD the Sine of the perpendicular Arc CA , as ER , the Sine of the Hypotenuse BE to EF , the Sine of the perpendicular Arc EH . W.W.D. PROPO

PROPOSITION XXIV.

The same Things being supposed, A Q, H K, the Sines of the Bases, are proportional to I A, G H, the Tangents of the perpendicular Arcs.

FOR after the same Manner, as in the last Proposition, we demonstrate that the Triangles Q A I, K H G, are equiangular; whence $QA : AI :: KH : HG$.

PROPOSITION XXV.

In a spherical Triangle A B C, right-angled at A, as the Cosine of the Angle B, at the Base B A, is to the Sine of the vertical Angle A C B, so is the Cosine of the Perpendicular to the Radius.

PREPARATION.

LET the Sides A B, B C, C A, be produced, so that B E, B F, C I, C H, be Quadrants; and from the Poles B and C, draw the great Circles E F D G, I H G, then will the Angles at E, F, I, H, be Right Angles. And so D is the Pole of B A E, (by Cor. 2. Prop. 2 of this,) and G the Pole of I F C B; also A E will be = Complement of the Arc B A, and F E the Measure of the Angle B = $\angle D$, and D F their Complement.: Also B C shall be = F I = Measure of the Angle G, and C F their Complement. Likewise C A = H D, and D C their Complement. These Things premised in the Triangles H I C, D C F,

The ELEMENTS of

DCF, right-angled at I and F, and having the same acute Angle C, since BA is less than a Quadrant, it will be as S, DF : S, HI :: S, DC : S, HC; that is, the Cosine of the Angle B, is to the Sine of the vertical Angle BCA, as the Cosine of CA is to Radius. W. W. D.

PROPOSITION XXVI.

The Cosine of the Base : Cosine of the Hypotenuse :: R : Cos. of the Perpendicular.

FOR in the Triangles AED, CFD, right-angled at E, F, having the same acute Angle D; because AE is less than a Quadrant, we have S, EA : S, CF :: S, DA : S, DC. W. W. D.

PROPOSITION XXVII.

*S, of the Base : R :: T, of the Perpendicular
T, of the Angle at the Base.*

FOR in the Triangles BAC, BEF, right-angled at A and E, and having the same acute Angle B; because AC is less than a Quadrant, we have S, BA : S, BE :: T, AC : T, EF. W. W. D.

PROPOSITION XXVIII.

Cos. of the vertical Angle : R :: T, of the Perpendicular : T, of the Hypotenuse.

IN the Triangles GIE, GHD, right-angled at I and H, and having the same acute

acute Angle G, because HD is less than HC,
or a Quadrant, it is as S, G H : S, G I ::
T, HD : T, IF.

PROPOSITION XXIX.

*S, of the Hypotenuse : R :: S, of the Perpen-
dicular : S, of the Angle at the Base.*

IN the aforesaid Triangles, we have S, IF :
S, G F :: S, HD : S, G D.

PROPOSITION XXX.

*R : Cos. of the Hypotenuse :: T, of the ver-
tical Angle : Cot. of the Angle at the Base.*

IN the Triangles HIC, DFC, right-angled
at I and F, and having the same acute
Angle C, because D F is less than a Qua-
drant, we have S, C I : S, C F :: T, H I :
T, D F, that is, R : Cos. B C :: T, C : Cot. B.

The last six Propositions are sufficient for
solving all the sixteen Cases of Right-angled
Spherical Triangles. These sixteen Cases, with
their Analogies deduc'd from the said Pro-
positions, are as follow :

F

Given

	Given besides the Right Angle.	Sought	The Proportions. <i>Vid. Fig. to Prop. 25, 26, 27, 28, 29, 30.</i>	
1	A C and C.	B	$R : \text{Cof. } CA :: S, C : \text{Cof. } B$ Of the same Kind with CA.	By the Inverse of Prop. 25.
2	A C and B.	C.	$\text{Cof. } CA : R :: \text{Cof. } B : S, C$ this is ambiguous.	by Prop. 25.
3	B and C.	A C.	$S, C : \text{Cof. } B :: R : \text{Cof. } CA$ of the same kind with the Angle B.	by Prop. 25 and 18.
4	B A, C A.	B C.	$R : \text{Cof. } BA :: \text{Cof. } AC : \text{Cof. } BC$. If BA, AC, be of the same Affection, and not Quadrants, then BC will be less than a Quadrant. If they be of a different Affection, BC shall be greater than a Quadrant.	by Prop. 26, 19, and 20.
5	B A, B C.	A C.	$\text{Cof. } BA : R :: \text{Cof. } BC : \text{Cof. } CA$. If BC be less than a Quadrant, then shall BA and CA be of the same Affection; if greater, of a different: but BA is given, and therefore the Species thereof: Wherefore the Species of AC is also given.	by Prop. 26 and 21.
6	B A, C A.	B.	$S, BA : R :: T, CA : T$ Of the same Affection with the opposite Side CA.	by Prop. 27 and 18.

Given

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	Given besides the Right Angle.	Sought	The Proportions.	
7	BA, B.	AC.	R: S, BA :: T, B: T, AC of the same kind with B.	by Prop. 27 and 18.
8	AC, B.	BA.	T, B: R :: T, CA: S, BA ambiguous.	by Prop. 27.
9	BC, C.	AC.	R: Cos. C :: T, BC: T, CA. If BC be less than a Qua- drant, the Angles C and B are of the same Affection, if greater, of a different. Therefore if the Species of the Angle B be given, then will AC be given.	By Prop. 28 and 21.
10	AC, C.	BC.	Cos. C: R :: T, CA: T, BC. And so if the Angle C, and CA, be of the same Affection, then BC shall be lesser than a Quadrant, if of a different, greater.	by Prop. 28, 20, 21.
11	BC, AC.	C.	T, BC: R :: T, CA: Cos. C. If BC be less than a Quadrant, then CA and BA, and consequently the Angles, shall be of the same Affection, if greater, of a different, but the Species of CA is given. Therefore the Species of the Angle C will be also given.	by Prop. 28, and 21.

	Given besides the Right Angle.	Sought	The Proportions.	
12	BC, B.	A C.	$R : S, BC :: S, B : S, AC$ of the same Species with B.	by Prop. 29, and 18.
13	AC, B.	B C.	$S, B : S, AC :: R : S, BC$ ambiguous.	by Prop. 29.
14	B C. A C.	B.	$S, BC : R :: S, AC : S, B$ of the same Species with CA.	by Prop. 29.
15	B, C.	B C.	$T, C : R :: \text{Cot. } B : \text{Cof. } BC$ And so if the Angles B and C are of the same Affection, then shall B C be less than a Quadrant, if of a different, greater.	by Prop. 30, 19, and 20.
16	BC, C.	B.	$R : \text{Cof. } BC :: T, C : \text{Cot. } B$ And so if B C be less than a Quadrant, the An- gles C and B shall be of the same Affection ; if greater of a different. But the Spe- cies of the Angle C is given ; therefore the Species of the Angle B will be given also.	by Prop. 30, and 31.

Of the Solution of Right-angled Spherical Triangles, by the five Circular Parts.

THE Lord *Napier*, (the Noble Inventor of Logarithms,) by a due Consideration of the Analogies, by which Right-angled Spherical Triangles are solv'd, found out two Rules, easy to be remembred, by means of which, all the sixteen Cases may be solved; for since, in these Triangles, besides the Right Angle, there are three Sides and two Angles; the two Sides comprehending the Right Angle, and the Complements of the Hypothenuse, and the two other Angles, were called by *Napier*, *Circular Parts*. And when there are given any two of the said Parts, and a third is sought; one of these three which is called the *Middle-Part*, either lyes between the other two Parts, which are called *Adjacent-Extremes*, or is separated from them, and then are called *Opposite-Extremes*; so if the Complement of the Angle *B* (*Fig. to Prop. 25.*) be supposed the middle Part, then the Leg *AB*, and the Complement of the Hypothenuse *BC*, are adjacent Extreme Parts; but the Complement of the Angle *C*, and the Side *AC*, are opposite Extremes. Also if the Complement of the Hypothenuse *BC*, be supposed the middle Part, then the Complements of the Angles *B* and *C* are adjacent Extremes, and the Legs *AB*, *AC*, are opposite Extremes. In like Manner, suppo-

ling the Leg AB the middle Part, the Complement of the Angle B and AC , are adjacent Extremes; for the Right Angle A does not interrupt the Adjacency, because it is not a Circular Part. But the Complement of the Angle C , and the Complement of the Hypotenuse BC , are opposite Extremes to the said middle Part. These Things premised.

R U L E. I.

In any Right-angled spherical Triangle, the Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Tangents of the adjacent Parts.

R U L E. II.

The Rectangle under the Radius, and the Sine of the middle Part, is equal to the Rectangle under the Cosines of the opposite Parts.

Each of the Rules have three Cases. For the middle Part may be the Complement of the Angle B , or C , or the Complement of the Hypotenuse BC ; or one of the Legs AB , AC .

Case 1. Let the Complement of the Angle C be the middle Part. Then shall AC , and the Complement of the Hypotenuse BC be adjacent Extremes. By *Prop. 28.* the Cosine of the Vertical Angle C is to Radius, as the Tangent of CA , is to the Tangent of the Hypotenuse BC . Then (by Alternation) w
sha

shall have $\text{Cos. } C : T, CA :: R : T, BC$.
 But $R : T, BC :: \text{Cot. } BC : R$, (as has been
 before shewn.) Wherefore $\text{Cos. } C : T, AC ::$
 $\text{Cot. } BC : R$; whence $R \times \text{Cos. } C = T,$
 $AC \times \text{Cot. } BC$.

And the Complement of the Angle B, and
 A B, are opposite Extremes, to the same mid-
 dle Part, the Complement of the Angle C.
 (and by *Prop. 25.*) as the Cosine of the An-
 gle C, to the Sine of the Angle C D F, so
 is the Cosine of D F to Radius. But the
 Sine of C D F $= S, AE = \text{Cos. } BA$, and Cos.
 $DF = S, EF = S, \text{Angle } B$. Whence it
 will be as $\text{Cos. } C : \text{Cos. } BA :: S, B : R$.
 And $R \times \text{Cos. } C = \text{Cos. } BA \times S, B$; that
 is, Radius drawn into the Sine of the mid-
 dle Part, is equal to the Rectangle under the
 Cosines of the opposite Extremes.

Case 2. Let the Complement of the Hy-
 pothenuse B C, be the middle Part, then the
 Complements of the Angles B and C, will
 be adjacent Extremes. In the Triangle D C F
 (by *Prop. 27.*) it is as $S, CF : R :: T, DF$.
 I, C. Whence (by Alternation) $S, CF :$
 $I, DF :: (R : T, C ::) \text{Cot. } C : R$. But
 $S, CF = \text{Cos. } BC$ and $T, DF = \text{Cot. } B$.
 Wherefore $R \times \text{Cos. } BC = \text{Cot. } C \times \text{Cot.}$
 B ; that is, Radius drawn into the Sine of the
 middle Part, is equal to the Product of the
 Tangents, of the adjacent extreme Parts.

And A B, A C, are the opposite Extremes
 to the said middle Part, viz. the Complement
 of B C; and (by *Prop. 26.*) $\text{Cos. } BA : \text{Cos.}$
 $BC ::$

$BC :: R : \text{Cof. } AC$. Wherefore we shall have $R \times \text{Cof. } BC = \text{Cof. } BA \times \text{Cof. } AC$.

Case 3. Lastly, Let AB be the middle Part; and then the Complement of the Angle B , and AC will be adjacent Extremes, (and by Prop. 27.) $S, AB : R :: T, CA : T, B$. Whence, $S, AB : T, CA :: (R : T, B ::)$ $\text{Cot. } B : R$. And so $R \times S, AB = T, CA \times \text{Cot. } B$.

Moreover, the Complements of BC , and the Angle C , are opposite Extremes to the same middle Part AB ; and in the Triangle GHD (by Prop. 25.) we have $\text{Cof. } D : S, DGH :: \text{Cof. } GH : R$. But $\text{Cof. } D = \text{Cof. } AE = S, AB$, and $S, G = S, IF = S, BC$. Also $\text{Cof. } GH = S, HI = S, C$. Wherefore it will be as $S, AB : S, BC :: S, C : R$. And hence $R \times S, AB = S, BC \times S, C$.

And so in every Case the Rectangle under the Radius, and the Sine of the middle Part, shall be equal to the Rectangle under the Cosines of the opposite Extremes, and to the Rectangle under the Tangents of the adjacent Extremes. And consequently if the aforesaid Equations be resolved into Analogies, (by 16. *El.* 6.) the unknown Parts may be found by the Rule of Proportion. And if the Part sought be the middle one; then shall the first Term of the Analogy be Radius, and the second and third, the Tangents or Cosines of the extreme Parts. If one of the Extremes be sought, the Analogy must begin with the other; and the Radius, and the Sine of the middle

middle Part, must be put in the middle Places; that so the Part sought may be in the fourth Place.

IN the Oblique-angled Spherical Triangle (*Fig. to Prop. 31.*) BCD, if a perpendicular Arc AC be let fall from the Angle C to the Base continued, if need be, so as to make two Right-angled Spherical Triangles BAC, DAC; then by those Right-angled Triangles, may most of the Cases of Oblique-angled ones be solved.

PROPOSITION XXXI.

The Cosines of the Angles B and D, at the Base BD, are proportional to the Sines of the Vertical Angles BCA, DCA.

FOR Cos. Angle B : S, BCA :: (Cos. CA : R ::) Cos. D : S, DCA. (*by 25. of this.*)

PROPOSITION XXXII.

The Cosines of the Sides BC, DC, are proportional to the Cosines of the Bases BA, DA.

FOR Cos. BC : Cos. BA :: (Cos. CA : R ::) Cos. DC : Cos. DA. (*by 26. of this.*)

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PROPOSITION XXXIII.

The Sines of the Bases BA, DA, are in a reciprocal Proportion of the Tangents of the Angles B and D at the Base BD.

BECAUSE (by 27. of this.) $S, BA : R :: T, AC : T.$ of the Angle B. And by the same inversely, $R : S, DA :: T,$ of the Angle D : $T, AC.$ Then will it be (by the Equality of perturbate Ratio, according to Prop. 23. El. 5.) $S, BA : S, DA :: T,$ Angle D : $T,$ Angle B.

PROPOSITION XXXIV.

The Tangents of the Sides BC, DC, are in a reciprocal Proportion of the Cosines of the Vertical Angles BCA, DCA.

BECAUSE by alternating the 28th Prop. we have
 $T, BC : R. :: T, CA : \text{Cos. } BCA,$
 and by the same $R : \text{Cos. } DCA :: T, DC : T,$
 $CA.$ Wherefore by Equality of perturbate Proportion.

$T, BC : \text{Cos. } DCA :: T, DC : \text{Cos. } BCA.$

PROPOSITION XXXV.

The Sines of the Sides BC, DC, are proportional to the Sines of the opposite Angles D and B.

BECAUSE by the 29th of this, $S, BC : R :: S, CA : S,$ of the Angle B. And
 by

by the same, inverting $R : S, DC : : S, \text{Angle } D : S, \text{ of } CA$, whence by Equality of perturbed Ratio, $S, BC : S, DC : : S, D : S, B$,

PROPOSITION XXXVI.

In any Spherical Triangle ABC, the Rectangle $CF \times AE$, or $FM \times AE$, contained under the Sines of the Legs BC, BA, is to the Square of the Radius, as IL or IA — LA the Difference of the Versed Sines of the Base CA, and the Difference of the Legs AM, to GN, the Versed Sine of the Angle B.

LET a great Circle PN be described from the Pole B; and let BP, BN be Quadrants, and then PN is the Measure of the Angle B; also describe from the same Pole B, a less Circle CFM thro' C; the Planes of these Circles shall be perpendicular to the Plane BON, (*by the 2d of this.*) And PG, CH, being perpendicular in the same Plane, fall on the common Sections ON, FM; suppose in G, H. Again draw HI, perpendicular to AO; and then the Plane drawn thro' CH, HI, shall be perpendicular to the Plane AOB. Whence AI which is perpendicular to HI will be perpendicular to the Right Line CI, (*by Def. 4. El. 11.*) and so AI is the Versed Sine of the Arc AC, and AL the Versed Sine of the Arc AM $\equiv BM - BA \equiv BC - BA$. The Isosceles Triangles CFM, PON, are equiangular, since MF, NO, as also CF, PO, (*by 16. El. 11.*) are parallel. Wherefore, if Perpendiculars CH, PG be drawn

drawn to the Sides FM, ON, the Triangles will be divided similarly, and we shall have $FM:ON::MH:GN$. Also, because the Triangles AOE, DIH, DLM, are equiangular, we shall have $AE:AO::IL:MH$.

But it has been proved that $FM:ON::MH:GN$. Wherefore it shall be as $AB \times FM:AO \times ON::IL \times MH:MH \times GN$, or as IL to GN , that is, the Rectangle under the Sines of the Legs, is to the Square of Radius, as the Difference of the Versed Sines of the Base, and the Difference of the Legs BC, BA, is to the Versed Sine of the Angle B. W. W. D.

PROPOSITION XXXVII.

The Difference of the Versed Sines of two Arcs drawn into half the Radius, is equal to the Rectangle under the Sine of half the Sum, and the Sine of half the Difference of those Arcs.

LET there be two Arcs BE, BF, whose Difference EF, let be bisected in D; then shall BD be the half Sum, and FD the half Difference of those Arcs. $GE=IL$ is the Difference of the Versed Sines of the Arcs BE, BF; also FO is the Sine of the half Difference of the Arcs. And because the Triangles CDK, FEG, are equiangular, we have $DK:GE::\left(\frac{1}{2}CD:\frac{1}{2}FE::\right)^{\frac{1}{2}}\frac{1}{2}CD:\frac{1}{2}FE$. Whence $DK \times \frac{1}{2}FE$, or $DK \times FO = GE \times \frac{1}{2}CD = IL \times \frac{1}{2}CD$, W. W. D.

PROPO

PROPOSITION XXXVIII.

The Versed Sine of any Arc, drawn into the half the Radius, is equal to the Square of the Sine of one half of the said Arc.

THE Triangles CBM, DEB are equiangular, since the Angles at M and E are Right Angles, and the Angle at B is common. Wherefore $EB : BD :: BM : BC$. And then will $EB \times BC = BM \times BD$; and $EB \times \frac{1}{2} BC = BM \times \frac{1}{2} BD = BM^2$. W. W. D.

PROPOSITION XXXIX.

In any Spherical Triangle ABC, whose Legs containing the Angle B are BC, AB, and Base subtending that Angle AC: If the Arc AM be taken = Difference of the Legs = $BC - AB$. Then shall the Rectangle under the Sines of the Legs BC, BA, be to the Square of the Radius, as the Rectangle under the Sine of the Arc $\frac{AC + AM}{2}$, and the Sine of the Arc $\frac{AC - AM}{2}$ is to the Square of the Sine of one half of the Angle B. Vid. Fig. to Prop. 36.

BECAUSE the Rectangle under the Sines of the Legs AB, BC, is to the Square of Radius, as IL is to the Versed Sine of the Angle

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Angle B, or as $\frac{1}{2} R. \times IL$ to $\frac{1}{2} R.$ drawn into the Versed Sine of the Angle B (by *Prop. 36* of this.) And since $\frac{1}{2} R. \times IL =$ Rectangle under the Sines of the Arcs $AC + AM$, and

$AC - AM$ (by *Prop. 37.* of this.) And all

$\frac{1}{2} R.$ drawn into the Versed Sine of the Angle B is equal to the Square of the Sine of one half of the Angle B. Therefore the Rectangle under the Sines of the Sides to the Square of Radius shall be as the Rectangle under the Sines of the Arcs $AC + AM$ and $AC - AM$, is

the Square of the Sine of one half the Angle B. W. W. D.



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The twelve Cases of Oblique-angled Spherical Triangles are as follow.

Given.	Sought.	make as
Angles B, D, and BC.	Angle C.	$R : \text{Cof. BC} :: T, B : \text{Cot. BCA}$, (by <i>Prop. 30.</i> of this.) Also $\text{Cof. B} : S, \text{BCA} :: \text{Cof. D} : S, \text{DCA}$ (by 31. of this.) Wherefore the Sum of the Angles BCA, DCA, if the Perpendicular falls within the Triangle, or the Difference, if it falls without, will be $= \text{BCD}$. Whether the Perpendicular falls within, or without the Triangle, may be known from the Affection of the Angles B and D (by 22. of this,) which Admonition ought to be observed in the following Solutions.
Angles B, C, D and the Side BC.	Angle A.	$R : \text{Cof. BC} :: T, B : \text{Cot. BCA}$ (<i>Prop. 30.</i> of this.) And $S, \text{BCA} : S, \text{DCA} :: \text{Cof. B} : \text{Cof. D}$ (by <i>Prop. 31.</i>) If BCA be less than BCD, the Angle D shall be of the same Affection with the Angle B. If BCA be greater than the Angle BCD, then the Angles B and D, shall be of a different Affection, by the Converse of <i>Prop. 22.</i> Given

	Given.	Sought	make as
3	The Sides B C, C D and the Angle B.	The Side B D.	R: Cof. B:: T, BC: T, BA (by 28, of this.) and Cof. BC: Cof. BA:: Cof. DC: Cof. DA. (by 32. of this.) the Sum and Difference of BA or DA according as the Perpendicular falls within, or without the Triangle, is equal to B D which cannot be known, unless the Species of the Angle D be first known.
4	The Sides B C, D B and the Angle B.	The Side C D.	R: Cof. B:: T, BC: T, BA (by 28 of this.) and Cof. BA: Cof. BC:: Cof. DA: Cof. DC. (by <i>Prop.</i> 32. of this) According as D A is similar or dissimilar to C A, or to the Angle B D C, so shall D A be less or greater than a Quadrant, (by 19. and 20 of this)
5	Angles B, D, and the Side B C.	The Side B D.	R: Cof. B:: T, BC: T, BA (by 28. of this.) And T, D: T, B:: S, BA: S, D A. (by 33. of this.) The Sum or Difference of which B A and D A is = B D.
6	The Sides B C, B D and the Angle B.	Angle D.	R: Cof. B:: T, BC: T, BA, (by <i>Prop.</i> 28. of this) and S, D A: S, BA:: T, B: T, D. (by 33. of this.) According as B D is greater or less than B A, the Angle D shall be similar or dissimilar to the Angle B. (by 22. of this)

Give

Given sought		make as
The Sides B C, D C, and the Angle B.	The Angle C.	$\text{Cof. BC} : R :: \text{Cot. B} : T,$ $\text{BCA, (by 30. of this.) and T,}$ $\text{DC} : T, \text{BC} :: \text{Cof. BCA} :$ $\text{Cof. DCA, (by 34. of this.)}$ The Sum or Difference of the Angles BCA, DCA, according as the Perpendicular falls within, or without the Triangle, is equal to the Angle BCD.
The Angles B, C, and Side B C.	The Side D C.	$\text{Cof. BC} : R :: \text{Cot. B} : T,$ $\text{BCA, (by 30. of this.) Also Cof.}$ $\text{DCA} : \text{Cof. BCA} :: T, \text{BC},$ $T, \text{DC, (by 34. of this.)}$ If the Angle DCA be similar to the Angle B (that is, if A D be similar to CA,) then DC shall be less than a Quadrant. If the Angles DCA and B be dissimilar, then DC shall be greater than a Quadrant, which follows (from <i>Prop. 18, 19, and 20. of this.</i>)
The Sides B C, D C and the Angle B.	The Angle D.	$S, CD : S, B :: S, BC : S,$ $D,$ which is Ambiguous. The Analogy follow from <i>Prop. 35. of this.</i>
The Angles B, D, and the Side B C.	The Side D C.	$S, D : S, BC :: S, B : S,$ $DC,$ which Side is Ambiguous.

Given.	Sought	make as
All the Sides A B, B C, C A. <i>Vid.</i> <i>Fig.</i> <i>Prop.</i> 36.	The Angle B.	As the Rectangle under the Sines of the Legs A B B C : The Square of Radius :: the Rectangle under A C + A M the Sines of the Arcs ———— $\frac{A C - A M}{2}$ and ———— : the Square of the Sine of $\frac{1}{2}$ the Angle B, (b) <i>Prop.</i> 39.
All the Angles G, H, D. <i>Vid.</i> <i>Fig.</i> 14.	The Side G D.	In the Triangle X N M, the Arc M N is the Complement of the Angle G H I to a Semicircle. X M is the Complement of the Angle G, and X N the Complement of the Angle D. And the Angle X, the Complement of the Side G D to a Semicircle. Wherefore if the Angles be changed into Sides, and the Sides into Angles, the Operation will be the same, as in <i>Case</i> 11. of this, since Arcs and their Complements to Semicircle have the same Sines.

The following REMARK by
SAMUEL CUN.

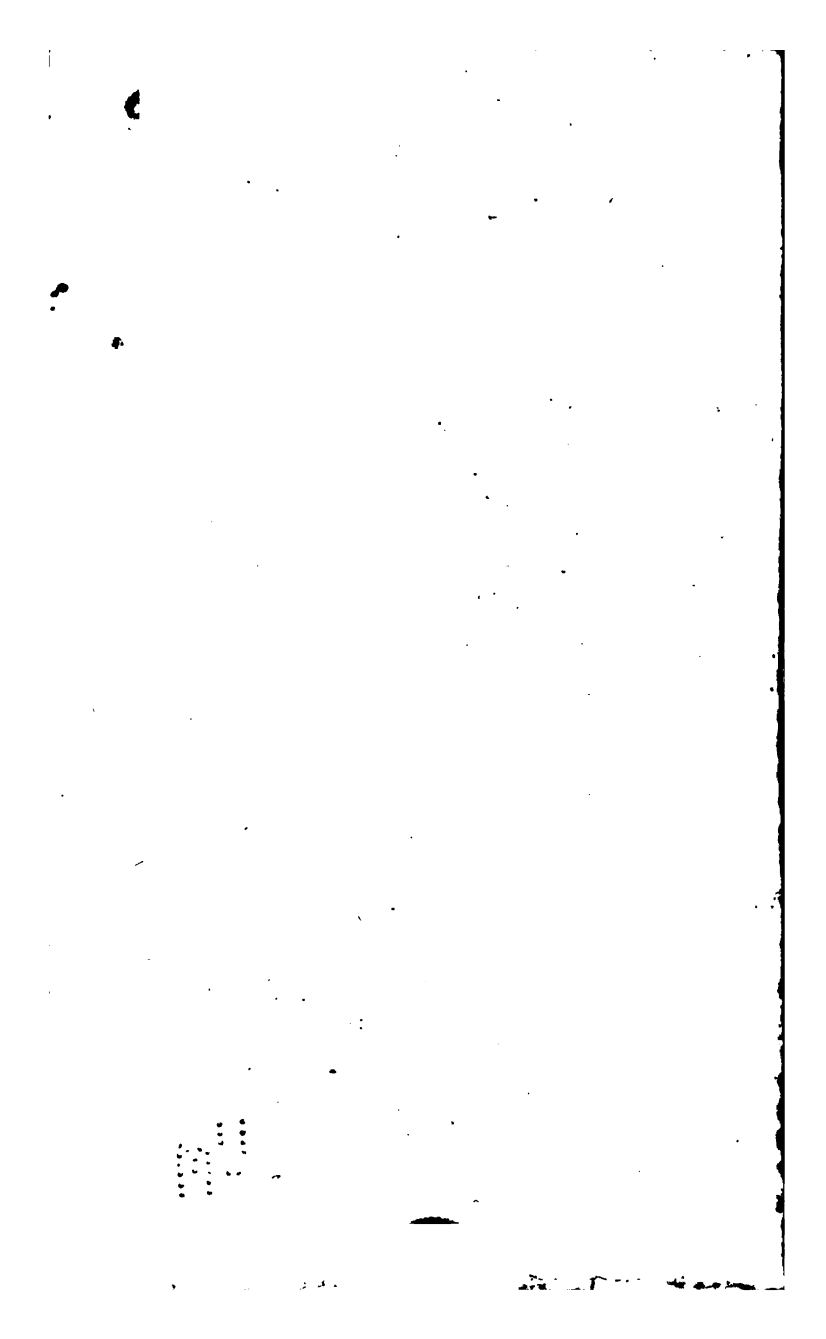
THAT this is true, but in a particular Case, viz. when two of the Angles of the Triangle are Right ones, and two of the Sides Quadrants, may be thus demonstrated. For if possible, let some Triangle RST , Fig. 14th be such, that its Sides RS , ST , TR , be equal to the Measures of GHD , HGD , GHD , the Angles of a Triangle GHD ; and also, that the Measures of ST , STR , TRS , the Angles of the Triangle RST , be equal to GH , GD , HD , the Sides of the Triangle GHD . And produce MX , MN , two Sides of the Supplemental Triangle to Semicircles, and they will meet some-where, suppose at E ; and there will be constructed thereby the Triangle NEX , of which XE (the Supplement of XM , which, by the 14th Prop. was the Supplement of the Measure of the Angle HGD) is equal to the Measure it self of the same Angle HGD : And in like manner, NE (the Supplement of NM , which, by the 14th Prop. was the Supplement of the Measure of the Angle GHD) is equal to the Measure it self of the same Angle GHD . But the third Side XN , is not the Measure of the third Angle GHD , but its Supplement, by the 14th Prop. Moreover, of the Angle ENX (whose Supplement is NXM)

H 2 the

the Measure, by the 14th *Prop.* is equal to $\angle G D$; and of the Angle $\angle X N E$, (whose Supplement is $\angle M N X$) the Measure, by the 14th *Prop.* is equal to $\angle H D$. But of the third $\angle N E X$, (which is equal to $\angle N M X$) the Measure is not equal to $\angle G H$, but its Supplement.

Now make $N V = R T = B K$, the Measure of the Angle $\angle G D H$, and draw the great Circle $E V$. And since $R S$, by Supposition is equal to the Measure of the Angle $\angle G H D$, which is equal to $\angle E N$; and since the Measure of the Angle $\angle S R T$, is by Supposition equal to $\angle D H$, which is also equal to the Measure of the Angle $\angle X N E$; the Angle $\angle X N E$, is equal to the Angle $\angle R$. Then consequently, by the 4th *Prop.* the Triangle $\triangle S R T$, $\triangle E N V$, will have the Base $S T$, equal to the Base $E V$; the Angle $\angle T$, to the Angle $\angle N V E$, and the Angle $\angle S$, to the Angle $\angle N E V$. But $S T$, (which is equal to $E V$), by Supposition, is equal to the Measure of the Angle $\angle H G D$; to which Measure $\angle X E$ is also equal. Therefore, $E V$ is equal to $\angle X E$; and consequently, by the 7th *Prop.* the Angle $\angle E V X$ is equal to the Angle $\angle E X V$; and the Angle $\angle E X V$ (whose Measure, as hath been shewn above, is equal to $\angle G D$) is equal to the Angle $\angle T$, (or $\angle N V E$), since by Supposition, the Measure of this is also equal to $\angle G D$. Therefore the Angle $\angle E V X$, is equal to the Angle $\angle E V N$, and so both Right ones; and consequently $\angle E X V$ a Right one also. Therefore, by the 2d *Cor.* to the 2d *Prop.* $E V$ and $E X$ are both Quadrants.

But



But if EV be a Quadrant, and at Right Angles to NX , then E , by 2d *Prop.* and its *Coroll.* is the Pole of NX ; and so EN a Quadrant also, and the Angle ENV a Right one. Therefore, if the Sides of a Triangle (NEV , or its Equal) RST , are equal to the Measures of the Angles of some other Triangle GHD , and the Measures of the Angles of the former, equal to the Sides of the latter; two Sides of such a Triangle RST , or GHD , must be Quadrants, and two Angles of each Right ones.

Therefore, if a Triangle RST be constructed, whose Sides are equal to the Measures of the Angles of another Triangle GHD : The Measures of the Angles of the Triangle RST , shall not be equal to the Sides of the Triangle GHD , unless in the one Case before-mentioned. Therefore the Measures of the Angles of the Triangle GHD , used as the Sides of a Triangle in the 11th Case, will not give us a Side of GHD , but the Measure of an Angle of the Triangle RST , unless in the one afore-mentioned Case; *which was to be demonstrated.*

But to find a Side GD of a Spherical Triangle GHD , whose Angles are all given, Produce MN , that Side of the supplemental Triangle, which is equal to the Supplement of the Measure of GHD , the Angle opposite to the Side sought, and MX , either of the other Sides till they meet, as in E . And here, as hath been before shewn, the Sides EX , EN , of the Triangle EXN , are exactly equal

equal to the Measures of the Angles HGD , GHD , of the Triangle GHD ; and of the Angles EXN , ENX , of the Triangle EXN , the Measures are equal to GD , HD . But the Side XN is equal to the Supplement of the Measure of the Angle $GIDH$. And of the Angle XEN , the Measure is equal to the Supplement of GH .

Therefore the SOLUTION is thus:

Change one of the Angles $G D H$, adjacent to the Side sought into its Supplement; and then work with the Measures of the Angles, as tho' they were Sides, and the Result will be GD , the Side sought.

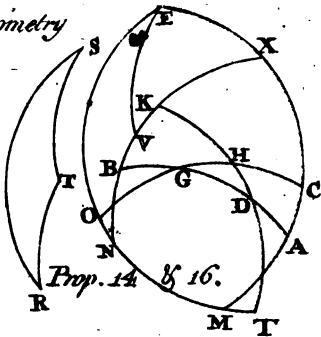
The preceding Fault, as well as the Omissions hereafter-mention'd, are not peculiar to our Author; but may be found in *Dr. Harris*, *Mr. Caswell*, *Mr. Heynes*, and many other Trigonometrical Writers.

In the Solution of our 8th and 9th Cases, they have told us, that the *Quæstæ* are Ambiguous; which sometimes, indeed, is true, but sometimes also false: Therefore, as I conceive it, they ought to have laid down Rules, by help of which, we might discover when the *Quæstæ* are Ambiguous, and when not.

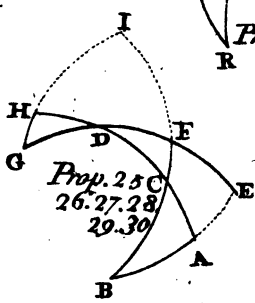
This Over-sight may be corrected by the following Directions: Wherein, because every Sine corresponds to two Arches, to one less than a Quadrant, and to another, which is the

63. Prop. 12.

10 13.



Prop. 14 & 16.



Prop. 25C
26. 27. 28.
29. 30.

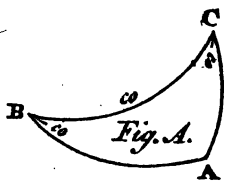
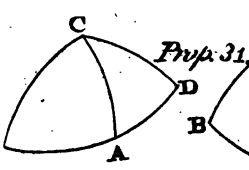
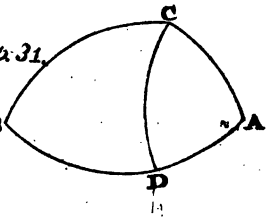


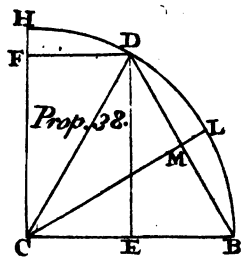
Fig. A.

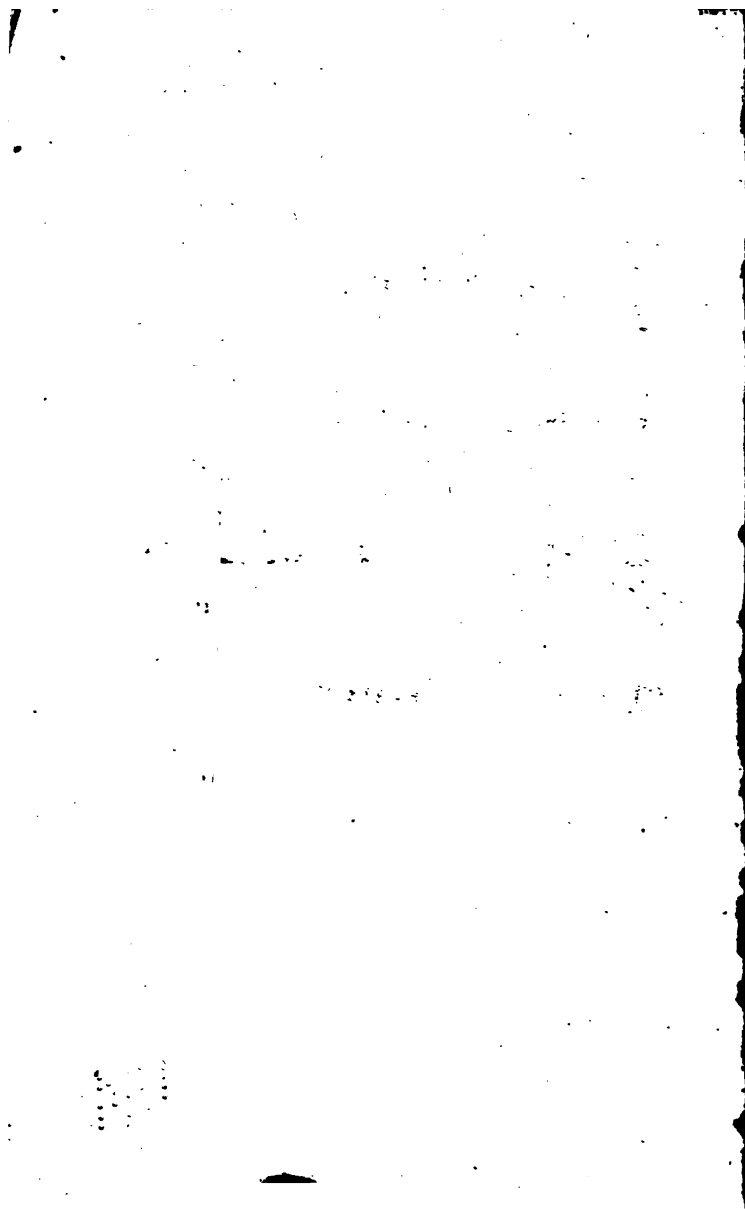


Prop. 31.



Prop. 32.





the Supplement of the former to a Semicircle, (a true Distinction of which, of these are to be used, being necessary to be known, before a proper Solution can be given, to such Problems as these are,) I shall beg leave, for brevity-sake, to call the lesser Arch the Acute Value, and the greater the Obtuse; whether the Sine be of an Angle, or a Side,

In the tenth Case, there are given two Angles B, D, and BC a Side opposite to one of those Angles D, to find DC the Side opposite to the other.

TO the Acute Value of DC, and also to its Obtuse one, add BC; and if each of these Sums are $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than a Semicircle, when the Sum of the Angles B, D, is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than two Right Angles; both the Values of DC may be admitted, and then is Ambiguous: But when only one of those Sums is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than a Semicircle, only one Value of DC can be true, viz. the $\left\{ \begin{smallmatrix} \text{Obtuse} \\ \text{Acute} \end{smallmatrix} \right\}$ one; and then is not Ambiguous.

In the ninth Case, there are given two Sides BC, DC, and one Angle B, opposite to DC one of those Sides, to find D the Angle opposite to the other.

TO the Acute Value of D, and also to its Obtuse Value, add B; and if each
of

of these Sums is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than two Right Angles, when the Sum of the Sides is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than a Semicircle ; both the Values of D may be admitted, and consequently, D is Ambiguous : But when only one of those Sums is $\left\{ \begin{smallmatrix} \text{greater} \\ \text{less} \end{smallmatrix} \right\}$ than two Right Angles, only one Value of D is true, *viz.* the $\left\{ \begin{smallmatrix} \text{Obtuse} \\ \text{Acute} \end{smallmatrix} \right\}$ one ; and then not Ambiguous.

Nor are we better used in the first Proposition ; for tho' it is determined by the given Angles, whether the Perpendicular falls within or without the Triangles, yet in each of those Varieties, the *Quæstia* will be sometimes Ambiguous, and sometimes not.

In this first Proposition, there are given two Angles B, D, and BC, a Side opposite to D one of them, to find C the third Angle.

1. Let the Perpendicular fall within ; that is, let the given Angles be of the same Species.

TO the Acute Value of DCA , and also to its Obtuse one, add the Angle BCA and if each of these Sums is less than two Right Angles, then either the Acute Value of DCA , or its Obtuse one added to BCA gives a Value of BCD ; which, therefore, is Ambiguous. And when only one of the

Sum

SPHERICAL TRIGONOMETRY.

65

Sums is less than two Right Angles, the Acute-Value of DCA , added to BCA , gives the only Value of BCD ; which then is not Ambiguous; tho' in both Varieties, the Perpendicular fell within.

2. Let the Perpendicular fall without; that is, let the given Angles be of different Species.

WHEN the Obtuse-Value of the Angle DCA is less than the Angle BCA , the Angle BCD may be had by subtracting either Value of DCA from BCA ; and then BCD is Ambiguous. But when the Obtuse-Value of DCA is not less than BCA , the Acute-Value of DCA , taken from BCA , gives the single Value of BCD ; which therefore is not Ambiguous; tho' in both Varieties, the Perpendicular fell without.

In the fifth Case, we lie under the same Misfortune, where there are given, as in the first, the Angles B, D , and the Side BC , to find BD , the Side lying between those given Angles.

1. When the Perpendicular falls within; that is, when the given Angles are of the same Species.

TO the Acute-Value of DA , and so also to its Obtuse one, add BA ; and if each of these Sums is less than a Semicircle, then either the Acute-Value of DA , or its
I
Obtuse

Obtuse one, added to BA , gives the Value of BD ; which thence is Ambiguous. And when only one of these Sums is less than a Semicircle, the Acute-Value of DA , added to BA , gives the only Value of BD ; which then is not Ambiguous; tho' in both Varieties, the Perpendicular fell within.

2. When the Perpendicular falls without; that is, when the given Angles are of different Species.

WHEN the Obtuse-Value of DA is less than BA , BD will be had by subtracting either Value of DA from BA ; and then BD is Ambiguous. But when the Obtuse-Value of DA is not less than BA , the Acute-Value of DA , taken from BA , leaves the only Value of BD ; which therefore is not Ambiguous; tho' in both Varieties, the Perpendicular fell without.

In the third, we have the same Omission; where there are given two Sides BC , CD , and B , an Angle opposite to CD one of them, to find the third Side BD .

FIRST, we may observe, that the Species of DA is always known; for it is of {the same } Affection with the Angle B ,
 {a different }
 when DC is {less } than a Quadrant,
 {greater }
 And, /

If AD be less than AB , and also the Sum of AD and AB less than a Semicircle; then
 AD

AD , either added to, or subtracted from AB , will give the Value of BD , which, therefore, is Ambiguous.

But if AD be not less than AB , or if their Sum be not less than a Semicircle; then their Sum, in the former, and their Difference in the latter Variety, shall give one single Value of BD , and then is not Ambiguous.

The seventh Case much resembles the third; for there are given two Sides BC , CD , and B , an Angle, opposite to CD one of them; to find the Angle BCD , lying between those two Sides.

AND here we may observe, that the Species of the Angle DCA is known; for it is of $\left\{ \begin{array}{l} \text{the same} \\ \text{a different} \end{array} \right\}$ kind with the Angle B , when DC is $\left\{ \begin{array}{l} \text{less} \\ \text{greater} \end{array} \right\}$ than a Quadrant. And,

If DCA be less than BCA , and the Sum of DCA and BCA , less than two Right-Angles; then, DCA either added to, or subtracted from BCA , will give the Angle BCD ; which therefore is Ambiguous.

If DCA be not less than BCA , or the Sum of DCA and BCA , not less than two Right-Angles; then their Sum in the
1 2
former,

former, and their Difference in the latter Variety, shall give the Single-Value of BCD which then is not Ambiguous.

N. B. If any one will be at the Trouble to make a double Calculation for the Side DC , or the Angle D , as taught in the Remarks on the 9th and 10th Cases, they will find the several Varieties in the 1st, 3d, 5th, and 7th, to be as here laid down in these easy Rules.

The Truth of these Rules may be easily deduced from the 10th, 13th, 18th, and 22d *Prop. of this*, and the 2d, 8th, and 13th *Examples*, following *Prop. 30. of this*.

In our third *Case* of Oblique-plain Triangles, our Author should have added this.

If AB be less than BC , the Angle A is Ambiguous, otherwise not.

Sam. Cunn's Observations on the
preceding Trigonometrical-
Tract.

[find our Author, as well as Dr. *Harris*,
Mr. *Caswell*, Mr. *Heynes*, and others of
the Trigonometrical-Writers, is mistaken in
some of the Solutions.

That the common Solution of the 12th
Case of Oblique-Spheric's is false, I have
demonstrated, and given a true one. See
Page 59.

In the Solution of our 9th and 10th Cases,
by other Authors called the 1st and 2d,
where are given and sought opposite Parts,
not only the afore-mention'd Authors, but
others, that I have met with, have told
us, that the Solutions are Ambiguous;
which Doctrine is, indeed, sometimes true,
but sometimes false; For sometimes the
Quæsitum is doubtful, and sometimes not;
and when it is not doubtful, it is sometimes
greater than 90 Degrees, and sometimes
less: And sure, I shall commit no Crime, if
I affirm, that no Solution can be given with-
out a just Distinction of these Varieties,
for the Solution of these Cases, see my
Directions at Pages 62, 63.

In the Solution of our 3d and 7th Cases,
other Authors reckon'd the 3d and 4th,
where

where there are given two Sides and an Angle opposite to one of them, to find the 3^d Side, or the Angle opposite to it ; all the Writers of Trigonometry, that I have met with, who have undertaken the Solutions of these two, as well as the two following Cases by letting fall a Perpendicular, which is undoubtedly the shortest and best Method for finding either of these *Quæstæ*, have told us, that the $\left\{ \begin{array}{l} \text{Sum} \\ \text{Difference} \end{array} \right\}$ of the Vertical Angles, or Bases, shall be the sought Angle or Side, according as the Perpendicular falls $\left\{ \begin{array}{l} \text{within} \\ \text{without} \end{array} \right\}$ which cannot be known, unless the Species of that unknown Angle, which is opposite to a given Side, be first known.

Here they leave us first to calculate that unknown Angle, before we shall know whether we are to take the Sum, or the Difference of the Vertical-Angles or Bases, for the sought Angle or Base : And in the Calculation of that Angle, have left us in the dark as to its Species ; as appears by my Observations on the two preceding Cases.

The Truth is the *Quæsitum* here, as well as in the two former Cases, is sometimes doubtful, and sometimes not ; when doubtful, sometimes each Answer is less than 90 Degrees, sometimes each is greater ; but sometimes one less, and the other greater, as in the two last mention'd Cases. When it is not doubtful, the *Quæsitum* is sometime greater

greater than 90 Degrees, and sometimes less. All which Distinctions may be made without another Operation, or the Knowledge of the Species of that unknown Angle, opposite to a given Side ; or which is the same thing, the falling of the Perpendicular, within or without. For which see my Directions at Pages 66, 67.

In the Solution of our 1st and 5th Cases, called in other Authors, the 5th and 6th ; where there are given two Angles, and a Side opposite to one of them, to find the 3d Angle, or the Side opposite to it ; they have told us, that the $\left\{ \begin{array}{l} \text{Sum} \\ \text{Difference} \end{array} \right\}$ of the Vertical Angles, or Bases, according as the Perpendicular falls $\left\{ \begin{array}{l} \text{within} \\ \text{without} \end{array} \right\}$ shall be the sought Angle or Side ; and that it is known whether the Perpendicular falls within or without, by the Affection of the given Angles.

Here they seem to have spoken, as though the *Quasitum* was always determin'd, and never Ambiguous ; for they have here determined, whether the Perpendicular falls within or without, and thereby whether they are to take the Sum, or the Difference of the Vertical-Angles or Bases, for the sought Angle or Side.

But, notwithstanding these imaginary Determinations, I affirm, that the *Quasitum* here, as in the two Cases last mentioned, is
some-

Sometimes Ambiguous, and sometimes not
and that too, whether the Perpendicular
falls within, or whether it falls without.
See my Solutions of these two Cases in
Page 64, 65.

The Determination of the 3d Case of
Oblique-Plain Triangles. See in Page 66.

SAM. CUNN





APPENDIX

By another Hand.

Containing the Elements of Astronomy, by the Solution of Spherical Triangles upon the Globe.



S **PHERICAL** Triangles are best represented by the Circles on, and those *appendant* to the *Globes*. The *Globe* being the *Original*, from whence all *Spherical* Trigonometry proceeds, and upon which the Sides and Angles of all *Spherical* Triangles are most *naturally* represented, and most *Expeditionously* measured.

Moreover, as the whole Art of *Navigation* whether by plain, *Mercator's* or *Wright's* Sailing is but the Solution and Application of the Doctrine of *Plain* Triangles: so *Astronomy* is the bare Solution and Application of the Doctrine of *Spherical* Triangles.

K

For

For as the Poet say's.

Cuncta Trigonus habet, Satagit quæ docta
(*Mathesis*)
Ille aperit Clausum quicquid Olympus habet.

In *Trigonometry* enclosed lye,
All the Grand Secrets of *Astronomy*.

But before the Practise, it will be necessary to premise some general *Definitions*, and necessary *Explanations* of the *Astronomical* Circles and *Terms* hereafter used.

First then.

1. A *Globe* is a round body, every part of whose Surface is equidistant from a point within it, called it's *Center*.

It may be conceived to be form'd by the revolution of a *Semicircle*, round it's *Diameter*. Every point in the Circumference of the generating *Semicircle* in it's circumvolution upon it's *Axis* or *Diameter*, marks out a Circle in the Surface of the *Globe*; whose Circles are greater or less as they are more remote from, or nearer to the Extreme points of the *Axis*, which are call'd it's *Poles*.

2. A *Great Circle* equally divides the *Globe* into two *Hemispheres* and has the same *Center* with it.

The

The most remarkable *Great Circles* are these the

Equator or *Equinoctial*.

Horizon.

Meridian.

Ecliptic.

Equinoctial Colure.

Solstitial Colure.

Azimuth or *Vertical Circles*.

3. The *Equator*, by Sailors emphatically called the *Line*, is equidistant from the Poles of the World, which are also it's Poles, it Divides the Earth into N°. and S°. Latitude, *on* it is reckon'd the *Longitude* of places, and *from* it their Latitude is counted: The *Equinoctial* is it's corresponding Circle in the Heavens, cuts always the *Horizon* in the East or West Points, and divides the *Celestial Globe* into N°. and S°. Declination: *on* it is reckoned *Right Ascension*, and from it the *Declination* of the ☉, ☿, or Stars, it is graduated from 10. 20. to 360, and represented in Plate 5. by the right line *Æ Q.*

4. The *Horizon* is a broad, wooden Circle, dividing the *Globe* into two Hemispheres; the *upper* and *visible*, which we see, from the lower and invisible, which we see not, *on* it are the points of the Compass, Days of the Month and Signs, by this Circle is determin'd the *Rising* and *Setting* of ☉, ☿ or Stars: *on* this we count their *Amplitude Azimuth* and ☉'s place every day in the Year: *from* it are counted the *Almicanters* or *Parallels of Al-*

titude; it's Poles are the *Zenith* or *Nadir*, *H X O*, in *Plate 5*, represents the *Horizon*.

5. The *Meridian* is a broad brass Circle, passing thro' the Poles of the *Horizon*, and Poles of the World; crossing both the *Equinoctial* and *Horizon* at right Angles, and divides the *Globe*, into the *Eastern* and *Western* Hemispherés, it's Poles lie in the E. and W. Points of the *Horizon*, in this *Meridian* and the *Horizon* the *Globe* hangs and turns, it's graduated into four Nineties, on it is reckoned the *Latitude* of places, the *Declination* and *Meridian* Altitude of ☉, ☿ or Stars, and from it the *Longitude* of places and hour from Noon, 'tis represented in *Plate 5*. by the outermost Circle, *Æ Z P H Q N S O*.

6. The *Ecliptic* cuts the *Equinoctial* at Angles of $23^{\circ}. 29'$; therefore it's Poles are distant from the Poles of the *Equinoctial* $23^{\circ}. 29'$. 'tis divided into 12 Signs each 30° . on either Side of the *Ecliptic*, about 8 Degrees off are two parallels which make the *Zodiac* or boundaries of the Planets. The *Ecliptic* divides the *Celestial Globe* into *N^o.* and *S^o.* Latitude of ☿ or Stars. On it is reckoned their *Longitude*, and from it their *Latitude* 'tis represented in *Plate 5*. by *⊗ A w*.

7. The *Equinoctial* Colure is that *Meridian*, which cutteth the *Equinoctial* at right Angles in those two Points, where the *Ecliptic* crosses the *Equinoctial* in γ and ϖ , and so passes thro' the Poles of the World, in *Plate 5*. 'tis represented by *P X S*.

8. The

8. The *Solstitial Colure* is that *Meridian* which cutteth the *Equinoctial Colure*, the *Equinoctial* and the *Ecliptic*, all three at right Angles, and passeth thro' the *Ecliptic*, where it touches both the *Tropic's* in $\S$ and $\mathcal{W}$. The utmost Circle in Plate 5. ÆZPHQNSO , represents it.

9. The *Azimuth* or *Vertical* Circles all pass thro' the *Zenith* and *Nadir* at right Angles to the *Horizon*: as the *Meridians* pass thro' the Poles of the World at right Angles, to the *Equinoctial*: that *Azimuth* Circle, which passes thro' the E. and W. points of the *Horizon* is called the *Prime Vertical* represented in Plate 5. by ZXN . The *Azimuths* are not drawn on *Globes*, but are represented by the *Quadrant* of *Altitude*, skrew'd on the *Zenith* which reaches to the *Horizon*, and is divided into 90° . for numbring the Degrees of *Altitude*, when the $\odot$, ☾ or Stars are not on the *Meridian*.

10. A *Smaller Circle* divides the *Globe* into two unequal Parts. The chief are the *Parallels* of.

Declination,
Altitude.
Latitude.

11. The *Parallels* of *Declination* are all those Circles on the *Globe*, which are drawn parallel to the *Equinoctial*. Four of which are most noted. The two *Tropic's* of $\S$ and

at $23^{\circ}. 29'$ from the *Equinoctial* both way and are the bounds of the $\odot$'s greatest *Declination* N° . and S° . markt in plate 5. ☿ ☿. Also the 2 *Polar Circles*, as far distant from the Poles of the World, as the *Tropics* are from the *Equinoctial*, that next the N Pole called the *Artic.* and that next the S Pole called the *Antartic Circle*; both which are bounds of the 2. *frigid Zones*, as the *Tropics* are of the *Torrid Zone*, betwixt the *Torrid* and the *Frigid* lie the two Temperate

12. The *Parallels of Altitude* are all those *Less Circles*, which are parallel to the *Horizon*, represented by the several Degrees of the *Quadrant of Altitude* moved about parallel to the *Horizon* called often *Almicanters*.

13. The *Parallels of Latitude* on the *Terrestrial Globe*, are the same as *Parallels of Declination* on the *Celestial*; both being parallel to the *Equinoctial*: but *Parallels of Latitude* on the *Celestial Globe*, are those *Smaller Circles* parallel to the *Ecliptic*, and represented by the degrees of the *Quadrant of Altitude*, skrew'd on the Poles of the *Ecliptic*, and moved parallel to the *Ecliptic*. The said *Quadrant* representing then the *Circles of Longitude* of the Stars.

14. The Principal Points are the Poles of the
 World, or of the *Equinoctial*.
 } *Horizon*.
 } *Meridian*.
 } *Ecliptic*.

Equi

{ *Equinoctial Colure.*
 { *Solstitial Colure.*
 { *Prime Vertical.*

15. The *Poles* of the *World*, or of the *Equinoctial* being the same, are the two points on which the *Globe* hangs and turns within the *Brazen Meridian* and *Wooden Horizon*: from one of which to the other, there passes a strait line thro' the Center of the *Globe* called the *Axis* of the *World*, in these two all *Meridians* meet: in *Plate 5*, they are markt *P.* and *S.*

16. The *Poles* of the *Horizon* are called the *Zenith* and *Nadir*, markt in *Plate 5*, *Z.* and *N.* the *Zenith* is that point in the *Heavens* directly over our *Heads*: and the *Nadir* is the contrary, directly under our *feet*, in these two do all *Azimuth* or *Vertical Circles* meet.

17. The *Poles* of the *Meridian* are the points of *East* and *West*, where the *Equinoctial* cuts the *Horizon* and are represented in *Plate 5*, by the point *X.* the other being invisible, being diametrically opposite.

18. The *Poles* of the *Ecliptic* are $23^{\circ}. 29'$ distant from the *Poles* of the *World*, in which two points the *Circles* of *Longitude* of the *Stars* do cross each other, which *Circles* are represented by the *Quadrant* of *Altitude* drew'd over the *Pole* of the *Ecliptic*.

19. The *Poles* of the *Equinoctial Colure* are *Æ. Q.* where the *Solstitial Colure* crosses the *Equinoctial*. And,

20. The

APPENDIX

20. The *Poles* of the *Solstitial Colure* is χ . where the *Equinoctial Colure* crosses the *Equinoctial* and *Ecliptic* in γ and ω , which four poles are Cardinal points, and divide the Year into 4 Quarters, the

Spring when $\odot$ appears in γ	
Summer	ϕ
Autumn	ω
Winter	χ

21. The two *Solstitial points* are ϕ and ω where the *Ecliptic* touches the 2 *Tropics* which the $\odot$ appears to do, when the days are longest or shortest, and causes them to stand still a while, without a Sensible lengthning or shortning, for the space of 10 or 12 days.

22. The *Poles* of the *Prime Vertical* are the N° . and S° . points of the *Horizon*, thro' which the *Brazen Meridian* passes.

23. The *Astronomical Terms* following which need Explanation are.

<i>Declination</i>	$\odot$'s <i>Position</i>
<i>Amplitude</i>	<i>Right Ascension</i>
<i>Altitude</i>	<i>Oblique Ascension</i>
<i>Azimuth</i>	and
<i>Latitude</i>	<i>Ascensional Difference</i>
<i>Longitude</i> .	

24. *Declination* is an Arch of the *Meridian* between the *Equinoctial* and the Star brought to the *Meridian*, and is either N° . or S° .

25. *Ampli*

25. *Amplitude* is an Arch of the *Horizon* between the East or West-points thereof, and the Star brought to the *Horizon*, and is either *Prime* when the Star rises, or *Occidial* when the Star sets.

26. *Altitude* is an Arch of the *Quadrant* of *Altitude*, between the *Horizon* and the Star, brought to the *Quadrant's* Edge, when skrew'd in the *Zenith*.

27. *Azimuth* is an Arch of the *Horizon* between the *Meridian*, and the Stars *Vertical*, represented by the *Quadrant* on the *Zenith*, when laid over the Star.

By Star, in these 4 last Definitions understand $\odot$, $\sphericalangle$, Star, or any point in the Heavens.

28. *Latitude* is either *Celestial* or *Terrestrial*. *Celestial* is the nearest distance of a Star from the *Ecliptic*, measur'd by a Circle of *Longitude* represented by the *Quadrant* of *Altitude*, laid over the Pole of the *Ecliptic* and the Star, and is either N°. or S°. as the Star is Situate.

Latitude on the *Terrestrial* Globe, is the nearest distance of a place from the *Equator*, measured by an Arch of the *Brazen Meridian*, between the *Equator* and place, when brought to the *Meridian*, which *Latitude* is always equal to the *Elevation* of the *Pole* above the *Horizon*: because it's Complement added to each, will make either of them *Exact Quadrants*.

drants. 'Tis also either N°. or S°. according to the Situation of the Place.

29. *Longitude* is either *Celestial* or *Terrestrial*. *Celestial*, as the *Longitude* of the ☉ is it's place in the *Ecliptic*. And *Longitude* of a Star, is an Arch of the *Ecliptic*, between γ and that *Circle of Longitude* which passes thro' that Star; represented by the Edge of the *Quadrant*, skrew'd over the Pole of the *Ecliptic*, and laid over the Stat.

Longitude of a place on the *Terrestrial Globe* is an Arch of the *Equator* between the first *Meridian*, and the *Meridian* that passes thro' the given place, represented by the *Brazen Meridian*, when the place is brought to it.

30. The Angle of the ☉'s *Position* is form'd by the Concurrence of two great Circles at the point of the ☉'s place in any Spherical Triangle.

31. *Right Ascension*, is an Arch of the *Equinoctial*, reckoning towards the East from γ . to that *Meridian*, which passes thro' the ☉ or Star.

32. *Oblique Ascension* is an Arch of the *Equinoctial*, reckoning towards the East, from γ . to that point, which rises with the ☉ or Star in an Oblique Sphere.

33. *Ascensional Difference* is the Difference in Degrees, between the *Right* and *Oblique Ascension*

Ascension aforefaid, which reduced into time, is the space between fix a Clock and ☉ Rise,

34. To reduce Degrees into Time, seeing 360° are Equal to 24 Hours. H.

For every $\left. \begin{array}{l} 15^\circ \\ 1 \\ .15' \\ 1 \end{array} \right\}$ allow $\left. \begin{array}{l} 1 \\ .4' \\ 1 \\ .4'' \end{array} \right\}$

35. Having Sun-rise, to find Sun-set, Length of Day, Length of Night, all the Year round,

Seeing we begin our Count at Mid-day, and Mid-night, the time of ☉ Rise, is consequently one half of the Night, and time of ☉ Set, one half of the Day: and the hours of both added together, must make up 12, wherefore subtract the hours of ☉ Rise from 12, the remains is the time of ☉ Set, and double the time of ☉ Set, gives the length of the Day, and double the time of ☉ Rise, gives the length of the Night all the Year round.

To Project the *Sphere* for the Latitude $51^\circ. 30' N^\circ$. Geometrically.

Q uarter the Circle, which represents the *Solstitial-Colure* or *Meridian*, with Z, N. the *Prime-Vertical*, and H O the *Horizon*: then set the *Chord* of the Latitude $51. 30$. from H to P, the *Pole* from Z, the *Zenith* to A, the *Equinoctial* from O to S, and from

N to Q, and draw $\overline{EQ}$ the *Equinoctial*, and PS the *Equinoctial-Colure*, from $\overline{AE}$ and Q on both sides, set off the *Chord* of 23° for the *Tropic's* $\overline{S}$ and $\overline{W}$, and draw $\overline{SXW}$ the *Ecliptic*; lay a Rule from $\overline{AE}$ to $\overline{S}$, which cuts PS in L, then thro' the 3 points $\overline{S}$ L $\overline{S}$, draw the *Tropic* of $\overline{S}$, which suppose the apparent Course of the *Sun* for that Day, which cuts the *Horizon* at $\odot$, thro' the 3 points P. $\odot$. S. draw the *Meridian* P $\odot$ S, which forms the 1st. and 3d. *Triangles* shaded, and at 6 a Clock, the $\odot$ appears at L, thro' Z. L. N. 3 Points, draw the *Vertical* Z L N, which forms the 4th shaded *Triangle*, X L L. Again, suppose the $\odot$ at M, thro' the 3 points Z. M. N. draw the *Azimuth* Z M N, and draw thro' P. M. S. the *Meridian* P M S, thus is formed both the 2d and 5th shaded *Triangles* X y z, and P Z M.

Case 1.

IN the Right-Angled Spherical-Triangle I H $\odot$ P, Fig. 1. Plate 5. Right-Angled at H are Given.

$\overline{PH}$ the Latitude $51. 30.$

$\overline{P \odot}$ the $\odot$'s Distance from the Pole $69. 30.$

{ H. $\odot$. the Complement of $\odot$'s Amplitude.

Required { H P $\odot$. the Hour of $\odot$'s Rising accounted from Midnight.

{ P $\odot$ H. the Angle of $\odot$'s Position, which the *Horizon* makes with the *Meridian*.

To

To represent this *Triangle* on the *Globe*.

Elevate the Pole to the given *Latitude*, and turn the *Globe* about, till $69^{\circ} \frac{1}{2}$ of the *Equinoctial-Colure* counted from the Pole, be at the *Horizon*, so will the *Triangle* be perfectly described by an *Arch* of the Brass *Meridian*, an *Arch* of the *Horizon*, and an *Arch* of the *Colure*.

The Complement of the $\odot$'s *Amplitude*, $\odot$ is numbred on the *Horizon*, from the *Colure* to the N° . point of the *Horizon*, and in this case is $57^{\circ} 34'$.

For the Angle $HP\odot$, count the degrees of the *Equinoctial*, which are contained between the *Colure* and the N° . side of the brass *Meridian*, which degrees reduced to time, *i. e.* $62^{\circ} 45'$. are equal to $4 h. 11'$ the hour from Midnight, the time of $\odot$ Rise, and $\frac{1}{2}$ the length of the Night, which subtract from 12, rests $7 h. 49'$ the time of $\odot$ set, and half the length of the Day.

For the Angle $P\odot H$, number on the *Colure*, and also on the *Horizon*, the Sides that include the Angle 90° . from the Angular Point, and the distance between those two points, where the reckoning ends, measured by the *Quadrant* of *Altitude*, or by *Compasses*, contains $56^{\circ} 39'$. the Angle of the $\odot$'s Position.

Case

Case 3.

IN the Right-Angled Spherical Triangle
 Xyz . Right-Angled at y , Fig. 2. Plate

Given $\begin{cases} \text{Xz. } \odot\text{'s Distance from } \gamma 72^\circ. \\ y\text{Xz. } \odot\text{'s greatest Declination } 23^\circ. \end{cases}$

Required $\begin{cases} \text{Zy. } \odot\text{'s present Declination.} \\ \text{Xy. } \odot\text{'s Right-Ascension.} \\ y\text{zX. } \odot\text{'s Position, or the Angle} \end{cases}$
 that the *Meridian* makes with the *Ecliptic*

To represent this Triangle on the Globe

Bring the $\odot$'s Place, which is $\pi 12^\circ$,
 the Brazen *Meridian*, and fix the Globe
 that Position, and the Triangle will be
 presented by an *Arch* of the *Equinoctial*,
Arch of the *Ecliptic*, and an *Arch* of the
 Brazen *Meridian*.

The $\odot$'s present *Declination* Xy . 22° .
 is measured on the Brazen *Meridian*, from
 the *Equinoctial*, to the $\odot$'s Place, and is so
 because betwixt the *Equinoctial* and N° . Po

The $\odot$'s *Right-Ascension* Xy , is measured
 on the *Equinoctial* $70^\circ \frac{2}{3}$ from γ , to the
 Brazen *Meridian*.

And for the Measure of the Angle yz .
 number 90° . on the *Meridian* from the
 place, and 90° on the *Ecliptic* from the
 place

ace, so shall the distance between these two
oints, measured by the *Quadrant* of *Alti-*
de, contain for the Angle $y z X$, $82^{\circ}. 20'$.

Case 3.

IN the Right-Angled *Spherical-Triangle*
A $\odot$ B, Right-Angled at B, Fig. 3.
late 5.

iven $\left\{ \begin{array}{l} \odot A B. \text{ the Complement of Lati-} \\ \text{tude } 38^{\circ}. 30'. \\ B \odot. \text{ the } \odot\text{'s Declination } 20. 30. N^{\circ}. \end{array} \right.$

quired $\left\{ \begin{array}{l} A \odot. \odot\text{'s Amplitude.} \\ A B. \text{ the Ascensional-Difference.} \\ A \odot B. \odot\text{'s Position at Rising.} \end{array} \right.$

To represent this *Triangle* on the *Globe*.

Depress the *Equinoctial* below the N° .
oint of the *Horizon* $38^{\circ}. 30'$ the Comple-
ment of the Latitude, then number $20^{\circ} \frac{1}{2}$
he $\odot$'s Declination on the *Equinoctial-Colure*,
nd bring that to the East-side of the *Hori-*
zon, so shall the *Triangle* be formed, and ap-
pear below the *Horizon* by A $\odot$, an *Arch* of
he *Horizon*. A B, an *Arch* of the *Equi-*
noctial. And B $\odot$, an *Arch* of the *Colure*.

A $\odot$ the $\odot$'s *Amplitude* from the *East*, is
numbered on the *Horizon* to the *Colure*, and
s $32^{\circ}. 26'$.

A B the *Ascensional-Difference*, is numbered
on the *Equinoctial*, from the *East-point* of
the

the *Horizon* to the *Colure*, and is $27^{\circ}.15'$ which reduced into time, is $1\ h. 49'$ before 6 for $\odot$ Rise *i. e.* $4. h. 11'$.

And the Angle of the $\odot$'s Position $A \odot B$ is found by continuing the Sides $\odot A$ and $\odot B$ to Quadrants, and then measuring betwixt those two Points, as in the first and 2d Cases.

Case 4.

IN the Right-Angled Spherical-Triangle AEL , Right-Angled at E , Fig. 4 Plate 5.

Given $\left\{ \begin{array}{l} AL \text{ the } \odot\text{'s Declination} = 20^{\circ}.30' N^{\circ} \\ LAE \text{ the Latitude} = 51.30 N^{\circ} \end{array} \right.$

Required $\left\{ \begin{array}{l} LE \text{ the } \odot\text{'s Altitude at Six.} \\ AE \text{ the } \odot\text{'s Azimuth from East.} \\ ALE \text{ the } \odot\text{'s Position or Angle} \end{array} \right.$ which the *Colure* makes with the Vertical at Six.

To represent this *Triangle* on the *Globe*.

Elevate the *Pole* to the given *Latitude*, fix the *Quadrant* of *Altitude* in the *Zenith*, and number the $\odot$'s *Declination* $20^{\circ} \frac{1}{2}$ on the *Equinoctial-Colure*, the *Colure* being brought to the East-point of the *Horizon*, bring the graduated *Edge* of the *Quadrant*, over the

$\odot$

☉'s *Declination* on the *Colure*, so will the Triangle be form'd by,

A L an *Arch* of the *Colure* = *Declination*.

A E an *Arch* of the *Horizon* = *Azimuth*, from the East.

L E an *Arch* of the *Quadrant* = ☉'s *Altitude*.

☉ A E, ☉'s *Azimuth* from E. is $13^{\circ} 6'$.

L E, ☉'s *Altitude* at 6. is $15^{\circ} 54'$.

and the Angle of the ☉'s Position A L E, is found by continuing the Sides L A and L E to Quadrants, and then measuring betwixt those two Points, as in the first and second Cases.

Of Oblique-Triangled Spherical-Triangles.

Case 1.

[N the *Oblique-Angled Spherical-Triangle* P Z M, Fig. 5. Plate 5. *Oblique-Angled* at Z:

Given	{	Z P the Complement of the Latitude	= $38^{\circ} 30'$
		M P the ☉'s Distance from the Pole	= $69. 30$
		M Z the Complement ☉'s Altitude	= $47. 20$

Required { P Z M the ☉'s *Azimuth* from the N°.
 { Z P M the hour of the Day from Noon.

E Z M P the ☉'s Angle of Position:
MTo

APPENDIX

To represent this *Triangle* on the *Globe*.

Elevate the *Pole* to the *Latitude*, then count $ZP\ 38^{\circ}. 30'$ the Complement of the *Latitude* from the *Pole*, on the S° . Side of the brazen *Meridian*, and where the Reckoning ends, skrew the *Quadrant* of *Altitude* i. e. on the *Zenith*, which is markt $51^{\circ}. 30'$ on the brass *Meridian*, this done, count the Side $MP\ 69^{\circ} \frac{1}{2}$ upon the *Equinoctial Colure* from the *Pole*, and the Side $MZ = 47^{\circ}. 20'$ on the *Quadrant* of *Altitude*, from the *Zenith* downwards, and so move the *Globe* and *Quadrant* of *Altitude* together, till the Numbers counted on the *Colure* and *Quadrant* of *Altitude*, meet in one Point, so will the *Triangle* be exactly delineated upon the *Globe*.

Now to find the several *Angles*.

For the *Angle Z*, count the Number of Degrees, of the *Horizon*, which are contain'd between the N° . Point of the *Meridian*, and the *Quadrant* of *Altitude* $115^{\circ}. 35'$ the $\odot$'s *Azimuth* from the N° .

For the *Angle Z P M*, the degrees of the *Equinoctial*, contain'd between the S° . side of the Brass *Meridian*, and the *Colure* 45° reduced into time, is 3 Hours from Noon, i. e. 9 in the Morning, or 3 Afternoon.

For the *Angle Z M P*, the *Angle* of Position, reckon 90° from the point *M*, on the *Circles*.

Circles M Z and M P, the distance betwixt those Points measured by the Quadrant of Altitude, applied to both, give $36^{\circ}.52'$. the Angle sought made by the *Meridian* and *Vertical*, concurring in M.

Also by having any 3 Parts of any Oblique-Triangle given, the other Three may be found, which afford no less Variety, than 60 Cases in an Oblique, and 30 in a Right.

As will appear, if we call the Latitude 1, ϕ 's Declination 2. Azimuth 3. Altitude 4. Position 5. Hour 6. Then may each of these be found from 10 different Data's.

	1	2	3	4	5	6
may be found from the Ten different Data's, directly underneath.						
1	1.3.4	1.5.6	1.7.6	1.7.6	2.3.4	2.3.4
2	2.3.5	1.4.6	1.4.6	2.3.5	1.4.6	2.3.5
3	2.3.6	1.4.5	1.4.5	2.3.6	2.3.6	1.4.5
4	2.4.5	1.3.6	2.4.5	1.3.6	1.3.6	2.4.5
5	2.4.6	1.3.5	2.4.6	1.3.5	2.4.6	1.3.5
6	2.5.6	1.3.4	2.5.6	2.5.6	1.3.4	1.3.4
7	3.4.5	3.4.5	1.2.6	1.2.6	1.2.6	3.4.5
8	3.4.6	3.4.6	1.2.5	1.2.5	3.4.6	1.2.5
9	3.5.6	3.5.6	1.2.4	3.5.6	1.2.4	1.2.4
10	4.5.6	4.5.6	1.2.3	1.2.3	1.2.3	1.2.3

Thus may we serve the five Parts of a Right Angled, Spherical Triangle.

Moreover, in the same Triangle, call the Side Z P, the distance of the Pole of the *Equinoctial*, from the Pole of the *Ecliptick*, then P M will be the Co-declination of a *Star* at M. Z M will be the Complement of the N° . Latitude of that *Star*, the Angle

M 2

M P Z,

APPENDIX.

MPZ, will be the Complement of the *Stars* Right-Ascension ; the Angle PZM, the *Stars-Longitude* from $\odot$, and PMZ the Angle of the *Stars* Position.

The same Triangle may be applied to Sundry other uses in *Geography*, *Navigation* and *Dialling*, for which we have no room in this small *Appendix*.

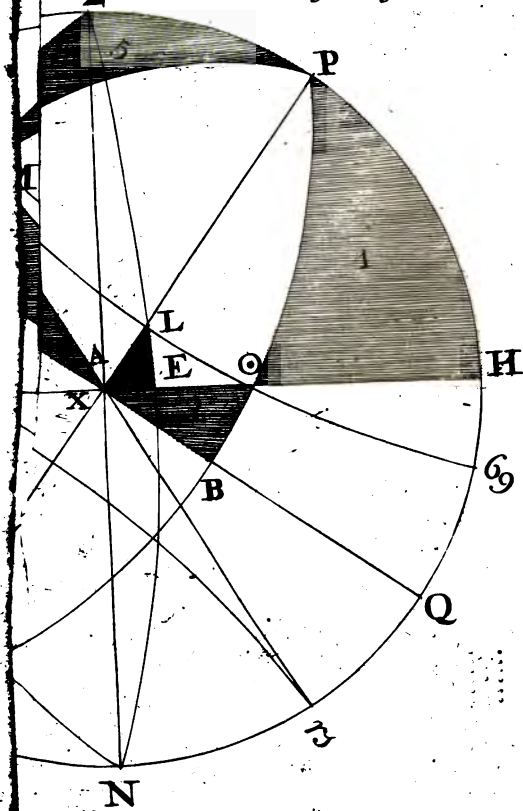
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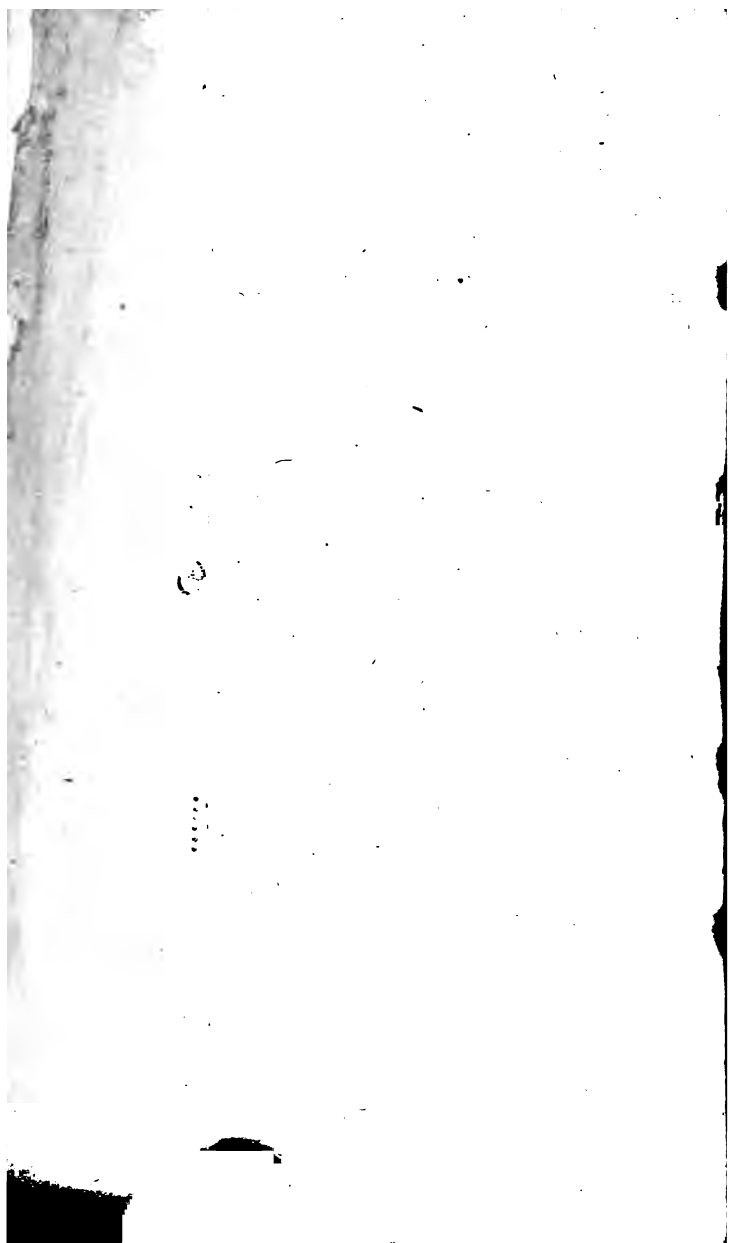
	For	Read
P. 24. li. 14.	AB : BC.	AB : AC.
p. 72. li. 8.	page 66.	p. 68.

Also omitted in the *London* Edition.

- p. 10. lin. 14. after *Sine* of, Read 2° to the Difference of the *Sines* of 13 and 17° . and so is the *Sine* of p. 33. lin. 15. after *Prop.*
2. Read and XB = Quadrant.

Place this over against Appendix







A Short

TREATISE

OF THE

NATURE and ARITHMETICK

OF

LOGARITHMS.



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THE PREFACE.



THE Mathematicks formerly received considerable Advantages; first, by the Introduction of the Indian Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all the Arts of Mathematicks. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for other Phenomena of Nature; and lastly, the Usurer computes the Interest of his Money.

The

P R E F A C E.

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom taking in the whole Doctrine, have indeed wrote learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouched. My Design therefore in the following Treatise, is to supply what seemed still wanting, viz. to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.

The wonderful Invention of Logarithms we owe to the Lord Neper, who was the first that constructed and published a Canon thereof, at Edinburgh, in the Year 1614. This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet Neper is universally allowed the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.

The same Lord Neper afterwards invented another and more commodious Form of Logarithms, which he communicated to Mr. Henry Briggs, Professor of Geometry at Oxford, who was hereby introduced as a Sharer in the completing thereof: But the Lord Neper dying, the whole Business remaining was devolved upon Mr. Briggs, who, with prodigious Application, and an uncommon Dexterity, compass'd a Logarithmic Canon, agreeable to that new Form for the first

P R E F A C E.

99

first twenty Chiliads of Numbers, (or from 1 to 20000) and for eleven other Chiliads, viz. from 90000 to 101000. For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publish'd at London in the Year 1624.

Adrian Vlacq published again this Canon at Gouda in Holland in the Year 1628, with the intermediate Chiliads before omitted, filled up according to Brigg's Prescriptions; but these Tables are not so useful as Brigg's, because the Logarithms are continued but to 10 Places of Figures.

Mr. Briggs also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to 15 Places of Figures, and has subjoined to them the Natural Sines, Tangents, and Secants, to 15 Places of Figures. The Logarithms of the Sines and Tangents are called Artificial Sines and Tangents, but the Sines and Tangents themselves are called Natural. These Tables, together with their Construction and Use, were publish'd after Brigg's Death, at London, in the Year 1633, by Henry Gellibrand, and by him called Trigonometria Britannica.

Since then, there have been published, in several Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms consist of but seven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 to 100000, which may be sufficient for most Uses.

The best Disposition of these Tables, in my Opinion, is that, first thought of by Nathaniel Roe, of Suffolk; and with some Alterations for

N

the

the better, followed by Sherwin in his Mathematical-Tables publish'd at London in 1705; wherein are the Logarithms from 1 to 101000 consisting of 7 Places of Figures. To which are subjoined the Differences and proportional Parts, by Means of which may be found easily the Logarithms of Numbers to 10000000, observing at the same Time that these Logarithms consist only of 7 Places of Figures. Here are also the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with some other Tables of Use in practical Mathematicks.





OF THE
NATURE and ARITHMETICK
OF
LOGARITHMS.



CHAP. I.

Of the ORIGIN and NATURE of Logarithms.



AS in Geometry, the Magnitudes of Lines are often defined by Numbers; so likewise on the other hand, it is sometimes expedient to expound Numbers by Lines, *viz.* by assuming some Line which may represent Unity, and the Double thereof; the Number 2, the Triple 3, the one half, the Fraction $\frac{1}{2}$, and so on. And thus the Genesis and Properties of some certain Numbers are better conceived, and

N 2

more

more clearly considered, than can be done by abstract Numbers.

* Fig. 1. Hence, if any Line a * be drawn into it self, the Quantity a^2 produced thereby, is not to be taken as one of two Dimensions, or as a Geometrical Square, whose Side is the Line a , but as a Line that is a third Proportional to some Line taken for Unity and the Line a . So likewise, if a^2 be multiplied by a , the Product a^3 , will not be a Quantity of three Dimensions, or a Geometrical-Cube, but a Line that is the fourth Term in a Geometrical-Progression, whose first Term is 1, and second a ; for the Terms 1, a , a^2 , a^3 , a^4 , a^5 , a^6 , a^7 , &c. are in the continual Ratio of 1 to a . And the Indices affixed to the Terms, shew the Place or Distance that every Term is from Unity. For Example, a^5 is in the fifth Place from Unity, a^6 is the sixth, or six times more Distant from Unity, than a , or a^2 , which immediately follows Unity.

If between the Terms 1 and a , there be put a mean Proportional which is $\sqrt{a}$, the Index of this will be $\frac{1}{2}$, for it's Distance from Unity will be one half of the Distance of a from Unity; and so $a^{\frac{1}{2}}$ may be written $\sqrt{a}$. And if a mean Proportional be put between a and a^2 , the Index thereof will be $1\frac{1}{2}$ or $\frac{3}{2}$, for it's Distance will be sesquialteral of the Distance of a from Unity.

If there be two mean Proportionals put between 1 and a ; the first of them is the Cube-Root of a , whose Index must be $\frac{2}{3}$, for that Term is distant from Unity only by a third Part of the Distance of a from Unity;
and

of LOGARITHMS. 101

and so the Cube-Root must be expressed by $\frac{1}{3}$. Hence, the Index of Unity is 0. for Unity is not distant from it self.

The same Series of Quantities, Geometrically Proportional, may be both ways continued, as well descending towards the Left Hand, as ascending towards the Right; for

the Terms $\frac{1}{a^5}, \frac{1}{a^4}, \frac{1}{a^3}, \frac{1}{a^2}, \frac{1}{a}, 1, a, a^2,$
 $a^3, a^4, a^5, \&c.$ are all in the same Geometrical Progression.

And since the Distance of a from Unity is towards the Right Hand, and positive or $+1$, the Distance equal to that on the contrary Side, *viz.* the Distance of

if the Term $\frac{1}{a}$ will be Negative or -1 ,

which shall be the Index of the Term $\frac{1}{a}$ for

which may be written a^{-1} . So likewise in the Term a^{-2} . The Index -2 shews that that Term stands in the second Place from Unity towards the Left Hand, and the Terms a^{-2} and $\frac{1}{a^2}$ are of the same Value. Also

a^{-3} is the same as $\frac{1}{a^3}$ For these negative Indices

shew that the Terms belonging to them, go from Unity the contrary way to that, by which the Terms whose Indices are positive do. These Things premised.

If on the Line AN, Fig. 2. both ways indefinitely extended, be taken, AC, CE, EG, GL, IL,

IL, on the right Hand. And also A r, r π , &c. on the left, all equal to one another. And if at the Points π , r, A, C, E, G, I, L, be erected to the Right Line A N, the Perpendiculars $\pi \kappa$, r Δ , A B, C D, E F, G H, I K, L M, which let be continually proportional and represent Numbers, whereof A B is Unity. The Lines A C, A E, A G, A I, A L—A π , respectively express the Distances of the Numbers from Unity, or the Place and Order that every Number obtains in the Series of Geometrical Proportionals, according as it is distant from Unity. So since A G is triple of the Right Line A C, the Number G H shall be in the third Place from Unity, if C D be in the first: So likewise shall L M be in the fifth Place, since A L = 5 A C. If the Extremities of the Proportionals κ , Δ , B, D, F, H, K, M, be joined by Right Lines, the Figure $\kappa \pi$ L M will become a Polygon consisting of more or less Sides, according as there are more or less Terms in the Progression.

If the Parts A C, C E, E G, G I, I L, be bisected in the Points c, e, g, i, l, and there be again raised the Perpendiculars c d, e f, g h, i k, l m, which are mean Proportionals between A B, C D; C D, E F; E F, G H; G H, I K; I K, L M; then there will arise a new Series of Proportionals, whose Terms beginning from that, which immediately follows Unity, are double of those in the first Series, and the Difference of the Terms are become less, and approach nearer to a Ratio of Equality than before. Likewise in this new Series, the

the Right Lines AL , AC , express the Distances of the Terms LM , CD , from Unity, *viz.* Since AL is ten times greater than AC , M shall be the tenth Term of the Series from Unity: And because AC is three times greater than AE , EF will be the third Term of the Series, if ED be the first; and there shall be two mean Proportionals between AE and EF , and between AB , and LM , there will be nine mean Proportionals.

And if the Extremities of the Lines Bd of FbH , &c. be joined by Right Lines, there will be a new Polygon made consisting of more, but shorter Sides than the last.

If again, the Distances AC , CE , CE , EE , &c. be supposed to be bisected, and mean Proportionals between every two of the Terms, be conceived to be put at those middle Distances; then there will arise another Series of Proportionals, containing double the Number of Terms from Unity than the former does; but the Differences of the Terms will be less; and if the Extremities of the Terms be joined, the Number of the Sides of the Polygon will be augmented according to the Number of Terms; and the Sides thereof will be less, because of the Diminution of the Distances of the Terms from each other.

Now in this new Series, the Distances AL , AC , &c. will determine the Orders or Places of the Terms; *viz.* if AL be five times greater than AC , and CD be the fourth Term of the Series from Unity, then LM will be the twentieth Term from Unity.

If

If in this Manner, mean Proportionals be continually placed between every two Terms the Number of Terms at last will be made so great, as also the Number of the Sides of the Polygon; as to be greater than any given Number, or to be infinite; and every Side of the Polygon so lessened, as to become less than any given Right Line; and consequently the Polygon will be changed into a Curv-lin'd Figure; for any Curv-lin'd Figure may be conceived as a Polygon, whose Sides are infinitely small and infinite in Number.

A Curve described after this Manner, is called *Logarithmical*; in which, if Number be represented by Right Lines standing at Right Angles to the Axis AN , the Portion of the Axis intercepted between any Number and Unity, shews the Place or Order that the Number obtains in the Series of Geometric Proportionals, distant from each other by equal Intervals. For Example, if AL be five times greater than AC , and there are a thousand Terms in continual Proportion from Unity to LM ; then will there be two Hundred Terms of the same Series from Unity to CD , or CD shall be the two hundredth Term of the Series from Unity; and let the Number of Terms from AB to LM be supposed what it will; then the Number of Terms from AB to CD , will be one fifth of the Number.

The Logarithmical-Curve may also be conceived to be described by two Motions, one of which is equable, and the other accelerated or retarded according to a given Ratio.

For Example, if the Right Line AB , moves uniformly along the Line AN , so that the End A thereof describes equal Spaces in equal Times; and, in the mean time, the said Line AB so increases, that the Increments thereof generated in equal Times, be proportional to the whole increasing Line, that is, if AB in going forward to cd , be encreased by the Increment od , and in an equal Time, when it is come to CD , the Increment thereof is Dp , and Dp to dc , is as do is to AB , that is, if the Increments generated in equal Times are always proportional to the Wholes; or, if the Line AB moving the contrary Way, diminishes in a constant Ratio, so that while it goes thro' the equal Spaces, $\Delta r, r \pi$ the Decrements $AB - \Gamma \Delta, \Gamma \Delta - \pi \Sigma$, are Proportionals to $AB, \Gamma \Delta$. Then the End of the Line increasing or decreasing in the said Manner, describes the Logarithmical-Curve: For since $AB:do::dc:Dp::DC:fq$, it shall be (by Composition of Ratio) as $AB:fc::dc:DC::DC:fe$. and so on.

By these two Motions, viz. the one equable, and the other proportionally accelerated or retarded, the Lord Neper laid down the Origin of Logarithms, and call'd the Logarithm of the Sine of any Arc, *That Number which nearest defines a Line that equally encreases, while, in the mean time, the Line expressing the whole Sine proportionally decreases to that Sine.*

It is manifest from this Description of the Logarithmic-Curve, that all Numbers at equal Distances are continually proportional. It is also plain, that if there be four Num-
O
bers

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bers $A B, C D, I K, L M$, such, that the Distance between the first and second, be equal to the Distance between the third and the fourth: Let the Distance from the second to the third be what it will, these Numbers will be proportional. For because the Distances $A C, I L$, are equal, $A B$ shall be to the Increment $D s$, as $I K$ is to the Increment $M T$. Wherefore (by Composition) $A B : D C :: I K : M L$. And contrariwise, if four Numbers be proportional, the Distance between the first and the second, shall be equal to the Distance between the third and the fourth.

The Distance between any two Numbers, is called the Logarithm of the Ratio of those Numbers, and indeed doth not measure the Ratio itself, but the Number of Terms in a given Series of Geometrical Proportionals, proceeding from one Number to another, and defines the Number of equal Ratio's, by the Composition whereof the Ratio of Numbers are known.

If the Distance between any two Numbers be double to the Distance between two other Numbers, then the Ratio of the two former Numbers shall be the Duplicate of the Ratio of the two latter. For let the Distance $I L$ between the Numbers $I K, L M$, be double to the Distance $A c$, between the Numbers $A B, c d$; and since $I L$ is bisected in l , we have $A c = I l = l L$; and the Ratio of $I K$ to $l m$, is equal to the Ratio of $A B$ to $c d$; and so the Ratio of $I K$ to $L M$, the Duplicate of the Ratio of $I K$ to $l m$, (by *Def. 10 El. 5.*) shall be the Duplicate of the Ratio of $A B$ to $c d$. Is

In like Manner, if the Distance EL be triple of the Distance AC , then will the Ratio of EF to LM , be triplicate of the Ratio of AB to CD : For because the Distance is triple, there shall be three times more Proportionals from EF to LM , than there are Terms, of the same Ratio, from AB to CD ; and the Ratio of EF to LM , as also of AB to CD , is compounded of the equal intermediate Ratio's, (by *Def. 5. El. 6.*) And so the Ratio of EF to LM , compounded of three times a greater Number of Ratio's, shall be triplicate of the Ratio of AB to CD ; So likewise if the Distance GL be quadruple of the Distance AC , then shall the Ratio of GH to LM , be quadruplicate of the Ratio of AB to CD .

The Logarithm of any Number, is the Logarithm of the Ratio of Unity to that Number, or it is the Distance between Unity and that Number. And so Logarithms express the Power, Place, or Order which every Number, in a Series of Geometrical-Progressions obtains from Unity. For Example, if there be 10.000.000 proportional Numbers from Unity to the Number 10, that is, if the Number 10 be in the 10.000.000th Place from Unity; then it will be found, by Computation, that in the same Series from Unity, to 2 there are 3.010.300 proportional Terms, that is, the Number 2 will stand in the 3.010.300th Place. In like Manner, from Unity to 3, there will be found 4.771.213 proportional Terms, which Number defines the Place of the Number 3. The Numbers

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10.000.000, 3.010.300, 4.771.213; shall be the Logarithms of the Numbers 10, 2, and 3.

If the first Term of the Series from Unity be called y , the second Term will be y^2 , the third y^3 . &c. And since the Number 10 is the 10.000.000th Term of the Series, then will $y^{10000000} = 10$. Also $y^{3910300} = 2$. Also $y^{4771213} = 3$; and so on.

Wherefore all Numbers shall be some Powers of that Number which is the first from Unity; and the Indices of the Powers are the Logarithms of the Numbers.

Since Logarithms are the Distances of Numbers from Unity, as has been shewn. The Logarithm of Unity shall be 0, for Unity is not distant from itself, but the Logarithms of Fractions are negative, or descending below nothing, for they go on the contrary Way. And so if Numbers increasing proportionally from Unity, have positive Logarithms, or such as are affected with the Sign +; then Fractions or Numbers in like Manner decreasing, will have negative Logarithms, or such as are affected with the Sign —; which is true when Logarithms are considered, as the Distances of Numbers from Unity.

But if Logarithms take their Beginning not from an integral Unit, but from a Unit, that is in some Place of Decimal Fractions. For Example, from the Fraction $\frac{1}{10.000.000.000}$, then all Fractions greater than this, will have positive Logarithms, and those that are less will have negative Logarithms. But more shall be said of this hereafter.

Since

Since in the Numbers continually proportional, D C, E F, G H, I K, &c. the Distances C E, E G, G I, &c. are equal, the Logarithms A C, A E, A G, A I, &c. of those Numbers shall be equidifferent, or the Differences of them shall be equal: And so the Logarithms of proportional Numbers are all in an Arithmetical-Progression; and from hence proceeds that common Definition of Logarithms, that Logarithms are Numbers, which being adjoined to Proportions, have equal Differences.

In the first Kind of Logarithms that *Neper* published, the first Term of the continual Proportionals was placed only so far distant from Unity, as that Term exceeded Unity; for Example, if $v n$ be the first Term of the Series from Unity A B, the Logarithm thereof, or the Distance A n , or B y , was, according to Him, equal to $v y$, or the Increment of the Number above Unity. As suppose $v y$ be 1,000.0001, he placed 0,000.000.1 for it's Logarithm A n ; and from hence, by Computation, the Number 10 shall be the 23.025.850th Term of the Series, which Number therefore is the Logarithm of 10 in this Form of Logarithms, and expresses it's Distance from Unity in such Parts whereof $v y$ or A n is one.

But this Position is entirely at Pleasure, or the Distance of the first Term may have any given Ratio to the Excess thereof above Unity, and according to that various Ratio (which may be supposed at Pleasure,) that is between $v y$ and B y , the Increment of the first Term above Unity, and the Distance of the same

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same from Unity, there will be produced different Forms of Logarithms.

This first Kind of Logarithms was afterwards changed by *Neper*, into another more convenient one, wherein he put the Number 10, not as the 23.025.850th Term of the Series, but the 10.000.000th; and in this Form of Logarithms, the first Increment v shall be to the Distance B y , or A z , as Unity; or AB , is to the Decimal-Fraction 0.4342.994 which therefore expresses the Length of the Subtangent AT . *Fig. 4.*

After *Neper's* Death, the excellent *Mrs. Henry Briggs*, by great Pains, made and published Tables of Logarithms according to this Form. Now since in these Tables, the Logarithm of 10, or the Distance thereof from Unity, is 10.000.000, and 1, 10, 100, 1000, 10.000, &c. are continual Proportionals, they shall be equidistant. Wherefore the Logarithm of the Number 100 shall be 2,0000000; of 1000, 3,0000000; and the Logarithm of 10.000 shall be 4,0000000; and so on.

Hence the Logarithms of all Numbers between 1 and 10, must begin with 0, or 0 must stand in the first Place to the Left-Hand; for they are less than the Logarithm of the Number 10, whose Beginning is Unity; and the Logarithms of the Numbers between 10 and 100 begin with Unity; for they are greater than 1,0000000, and less than 2,0000000. Also the Logarithms between 100 and 1000, begin with 2, for they are greater than the Logarithm of 100, which begins with 2, and less than the Logarithm

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of a 1000 that begins with 3. In the same Manner it is demonstrated, that the first Figure to the Left-Hand of the Logarithms between 1000 and 10.000, must be 3; and the first Figure to the Left-Hand of the Logarithms between 10.000 and 100.000, will be 4; and so on.

The first Figure of every Logarithm to the Left-Hand, is called the Characteristic or Index, because it shews the highest or most remote Place of the Number from the Place of Units. For Example, if the Index of a Logarithm be 1, then the highest or most remote Place from Unity of the correspondent Number to the Left-Hand, will be the Place of Tens. If the Index be 2, the most remote Figure of the correspondent Number shall be in the second Place from Unity, that is, it shall be in the Place of Hundreds; and if the Index of a Logarithm be 3, the last Figure of the Number answering to it, shall be in the Place of Thousands. The Logarithms of all Numbers that are in Decuple or Subdecuple Progression, only differ in their Characteristics, or Indices, they being written in all their Places with the same Figures. For Example, the Logarithms of the Numbers 17, 170, 1700, 17.000, are the same, unless in their Indices; for since 1 is to 17, as 10 to 170, and as 100 to 1700, and as 1000 to 17.000; therefore the Distances between 1 and 17, between 10 and 170, between 100 and 1700, and between 1000 and 17.000, shall be equal. And so since the Distance between 1 and 17, or the Logarithm of the Number

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17 is 1.2304489, the Logarithm of the Number 170, will be = 2.2304489, and the Logarithm of the Number 1700 shall be 3.2304489, because the Logarithm of the Number 100 = 2.0000000. In like Manner, since the Logarithm of the Number 1000 = 3.0000000, the Logarithm of the Number 17.000 shall 4.2304489.

So also the Numbers, 6748. 674. 8. 67. 48. 6. 748.0, 6748. 0, 06748, are continual Proportionals in the Ratio of 10 to 1; and so their Distances from each other shall be equal to the Distance or Logarithm of the Number 10, or equal to 1.0000000.

And so since the Logarithm of the Number 6748 is 3.8291751, the Logarithms of the other Numbers shall be as in the Margin; where you may observe that the Indices of the last two Logarithms are only negative, and the other Figures positive; and so when those other Figures are to be added, the Indices must be subtracted, and contrariwise.

6748	3,8291751
674,8	2,8291751
67,48	1,8291751
6,748	0,8291751
0,6748	—1,8291751
0,06748	—2,8291751

C H A P. II.

THE ARITHMETICK

CHAP. II.

*Of the Arithmetick of Logarithms
in whole Numbers, or whole Num-
bers adjoined to Decimal Fracti-
ons. Fig. 2.*



BECAUSE, in Multiplication, Unity is to the Multiplier, as the Multiplicand is to the Product, the Distance between Unity and the Multiplier, shall be equal to the Distance between the Multiplicand and the Product; if therefore, the Number GH be to be multiplied by the Number EF , the Distance between GH and the Product, must be equal to the Distance AE , or to the Logarithm of the Multiplier; and so if GL be taken equal to AE , the Number LM shall be the Product, that is, if to the Logarithm of the Multiplicand AG be added, the Logarithm of the Multiplier AE , the Sum shall be the Logarithm of the Product.

In Division, Unity is to the Divisor, as the Quotient is to the Dividend, and so the Distance between the Divisor and Unity, shall be equal to the Distance between the Dividend and the Quotient. So if LM be to be divided by EF , the Distance EA shall be

P
equal

equal to the Distance between LM and the Quotient, and so if LG be taken equal to EA , the Quotient will be at G ; that is, if from AL , the Logarithm of the Dividend, be taken GL , or AE , the Logarithm of the Divisor, there will remain AG , the Logarithm of the Quotient.

And from hence it appears, that whatsoever Operations in common Arithmetick are performed by multiplying or dividing of great Numbers, may be much easier, and more expeditiously done by the Addition or Subtraction of Logarithms.

Let, for Example, the Number 7589 be to be multiplied by 6757. Now, if the Logarithms of those Numbers

be added together, as in	Log. 3. 8801846
the Margin their Sum	Log. 3. 8297539.
will be the Logarithm	Log. 7. 7099385.
of the Product, whose	

Index 7 shews that there are seven Places of Figures, besides Unity, in the Product; and in seeking this Logarithm in Tables, or the nearest equal to it, I find that the Number answering thereto, which is less than the Product is 51278000, and the Number greater than the Product is 51279000, and if the adjoined Differences and proportional Parts be taken, the Numbers that must be added to the Place of Hundreds and Tens in the Product are 87, and that which must be added in the Place of Unity, will necessarily be 3, since seven times nine = 63, and so the true Product shall be 51278873. If the Index of the Logarithm had been 8 or 9, then the Numbers

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Numbers to be added in the Place of Hundredths or Tenths, could not be had from those Tables of Logarithms, which consist of but 7 Places of Figures, besides the Characteristick, and so in this Case, the *Vlacquian* or *Briggian* Tables should be used; in the former of which, the Logarithms are all to ten Places of Figures, and in the latter to fourteen.

If the Number 78956 be to be divided by 278, by subtracting the Logarithm of the Divisor from the Logarithm of the Dividend, the Logarithm of the Quotient will be had. And to this Logarithm, the Number 282, 719 answers; which therefore shall be the Quotient.

$$\begin{array}{r} \text{Log. 4. } 8954004 \\ \text{Log. 2. } 4440448 \\ \hline \text{Log. 2. } 4513556 \end{array}$$

Because Unity, any assumed Number, the Square thereof, the Cube, the Biquadrate, &c. are all continual Proportionals, their Distances from each other shall be equal to one another. And so it is manifest, the Distance of the Square from Unity, is double of the Distance of its Root from the same: Also the Distance of the Cube, is triple of the Distance of its Root; and the Distance of the Biquadrate, is quadruple of the Distance of its Root from Unity, &c. And so if the Logarithm of any Number be doubled, we shall have the Logarithm of its Square; if it be tripled, we shall have the Logarithm of its Cube, and if it be quadrupled, the Logarithm of its Biquadrate. And contrariwise, if the Logarithm of any Number be bisected, we shall have the Lo-

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garithm of the Square-Root thereof: Moreover, a third Part of the said Logarithm, will be the Logarithm of the Cube-Root of the Number; and a fourth Part, the Logarithm of the Biquadrate-Root of that Number.

Hence, the Extraction of all Roots are easily performed, by dividing a Logarithm into as many Parts as there are Units in the Index of the Power. So if you want the Square-Root of 5, the half of 0,6989700 must be taken, and then that half 0.3494850 will be the Logarithm of the Square-Root of 5, or the Logarithm of $\sqrt{5}$, to which the Number 2.23606 nearly answers.



CHAP. III.

Of the Arithmetick of Logarithms, when the Numbers are Fractions. Fig. 3.



WHEN Fractions are to be worked by Logarithms, it is necessary, for avoiding the Trouble of adding one Part of a Logarithm, and subtracting the other, that Logarithms do not begin from an integral Unit, but from some Unit that is the tenth or hundredth

dredth Place of Decimal-Fractions : For Ex-

1

ample, let P O be $\frac{1}{10000000000}$ and from this let the Logarithms begin. Now this Fraction is ten times more distant from Unity to the Left-Hand, than the Number 10 is distant there from to the Right; for there are 10 proportional Terms in the Ratio of 10 to 1, from Unity to P O. And so if A B be Unity, the Logarithm thereof, according to this Supposition, will not be 0, but O A will be $\equiv 10.00000000$; for the Distance of any Tenth from Unity is 1.00000000 , whence the Distance of the Number 10 from P O will be 11.00000000 . Also the Distance of the Number 100 from P O, or it's Logarithm, beginning from P O, shall be 12.00000000 , and the Logarithm of 1000, or the Distance from P O, will be 13.00000000 . And thus, the Indices of all Logarithms are augmented by the Number 10; and those Fractions whose Indices are — 1, or, — 2, or — 3, &c. are now made 9, 8, or 7, &c.

But if Logarithms begin from the Place of a Fraction, whose Numerator is Unity, and Denominator Unity with 100 Cyphers added to it, (which they must do when Fractions occur that are less than P O) then that Fraction will be 100 times more distant from Unity, than 10 is distant from it; and so the Logarithm of Unity will have 100 for the Index thereof. And the Logarithm of any Tens will have 101 for the Index, that of any Hundreds 102, and so on; all the Indices being augmented by the Number 100.

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The Logarithms of all Fractions that are greater than PO (whereat they begin) will be positive. And since the Numbers 10, 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. are in a continued Geometrical Progression, they will be equally distant from each other; and accordingly their Logarithms will be equidifferent: And so when the Logarithm of 10, is 11.0000000, and the Logarithm of Unity is 10.0000000, and the Logarithm of the Fraction $\frac{1}{10}$ will be 9.0000000, and the Logarithm of the Fraction $\frac{1}{100}$ will be 8.0000000, and in like manner, the Index of the Logarithm of $\frac{1}{1000}$ will be 7. Also for the same Reason, if the Index of the Logarithm of Unity be 100, and of 10 be 101, then will the Index of the Logarithm of the Fraction $\frac{1}{10}$ be 99, and the Index of the Logarithm of $\frac{1}{100}$ will be 98, and the Index of Logarithm of the Fraction $\frac{1}{1000}$ shall be 97, &c. And these Indices shew in what Place from Unity, the first Figure of the Fraction, not being a Cypher, must be put. For Example, if the Index be 4, the Distance thereof from the Index of Unity, (which is 10) viz. 6, shews that the first significative Figure of the Decimal, is in the sixth Place from Unity; and therefore, five Cyphers are to be prefixed thereto towards the Left-Hand. So also if the Index of Unity be 100, and the Index of the Fraction be 80, the first Figure thereof shall be in the 20th Place from Unity, and 19 Cyphers are to be prefixed thereto.

Now, let it be required to multiply the Fraction GH by the Fraction DC. Because Unity is to the Multiplier, as the Multiplicand

multiplicand is to the Product; the Distance between
 Unity and the Multiplier shall be equal to
 the Distance between the Multiplicand and
 the Product. Therefore, if there be taken
 $GI = AC$, the Product IK shall be at I .
 And accordingly, if from OG , the Loga-
 rithm of the Multiplicand, there be taken GI
 or AC , there will remain OI , the Loga-
 rithm of the Product. But $AC = OA -$
 OC , which taken from OG , there will re-
 main $OG + OC - OA = OI$, that is, if
 the Logarithm of the Multiplier and Mul-
 tiplicand be added together, and from the
 Sum be taken the Logarithm of Unity, (which
 is always expressed by 10 or 100 with Cy-
 phers) the Logarithm of the Product will
 be had. For Example, let the Decimal-Fracti-
 on 0,00734 be to be multiplied by the Fracti-
 on 0,000876. Set down 100 for the Index
 of the Logarithm of Unity, and then the
 Logarithms of the Fractions will be as in the
 Margin, which being added together, and
 the Logarithm of Unity being
 taken away from the Sum, the
 Remainder is the Logarithm
 of the Product, whose Index 94
 shews that the first Figure of
 the Product is in the sixth Place from Unity,
 and so there must be five Cyphers prefixed,
 and then the Product will be .00000642984

In Division the Divisor is to Unity, as
 the Dividend is to the Quotient; and so the
 Distance between the Divisor and Unity shall
 be equal to the Distance between the Divi-
 dend and the Quotient. And so if the Fracti-
 on

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on IK be to be divided by DC , you must take $IG = CA$, and the Place of the Quotient shall be G . But $CA = OA - OC$, which being added to OI , we have $OA + OI - OC = OG$, that is, if the Logarithm of Unity be added to the Logarithm of the Dividend, and from the Sum be taken the Logarithm of the Divisor, there will remain the Logarithm of the Quotient; so if the Number CD be to be divided by IK , you must take the Distance $CS = IA$, and then ST will be the Quotient, whose Logarithm is $OA + OC - OI$. Let $CD = 0.347$, $IK = 0.00478$. Then add the Logarithm of Unity to the Logarithm of CD , that is, put 1 or 10 before the Index thereof, and from that subtract the Logarithm of the Divisor, and the Remainder will be the Logarithm of the Quotient, whose Index 11. sheweth that the Quotient is between the Numbers 10 and 100; and I seek the Number answering the Logarithm, which I find to be 75.549. If the Logarithm of a Vulgar-Fraction for Example, $\frac{7}{8}$ be required, the Logarithm of Unity must be added to the Logarithm of the Numerator 7, or which is all one, you must put 10 or 100 before the Index thereof, and subtract from it the Logarithm of the Denominator 8, there will remain the Logarithm of the Vulgar-Fraction $\frac{7}{8}$, or the Decimal .875

19. 54032
7. 67942
11. 86090

10. 84509
0. 90309
9. 94200

If the Powers of any Fraction DC be required, you must assume EC, EG, GI, IL, each equal to AC; and then EF will be the Square, GH the Cube, and IK the Biquadrate of the Number DC; for they are continually Proportional from Unity. Besides, $AE = 2 AC = 2 AO - 2 OC$, whence $OE = OA - AE = 2 OC - AO$, that is, the Logarithm of the Square is the Double of the Logarithm of the Root, less the Logarithm of Unity. In like Manner, since $AG = 3 AC = 3 OA - 3 OC$, we shall have $OG = OA - AG = 3 OC - 2 OA =$ the Logarithm of the Cube = triple the Logarithm of the Root less the Double of the Logarithm of Unity. For the same Reason, because $AI = 4 AC = 4 OA - 4 OC$, we have $OI = 4 OC - 3 OA$, which is the Logarithm of the Biquadrate. And universally, if the Power of a Fraction be n , and the Logarithm L , then shall the Logarithm of the Power $n = nL - nOA + OA$, that is, if the Logarithm of a Fraction be multiplied by n , and from the Product be taken the Logarithm of Unity, multiplied by $n - 1$, the Logarithm of the Power n of that Fraction will be had.

For Example, if it is required to find the 6th Power of the Fraction $\frac{1}{2} = .5$ the Logarithm of this Fraction is 8.6989700, which being multiplied by 6; gives the Number 52.938200; and if from 52 the Number 50, which is the Index of the Logarithm of Unity drawn into 5, be taken away, the Remainder will be the Logarithm of the 6th Power, viz.

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2. 1938200, to which the Number, 0000000-15625 answers. For the Index 2 shews that 7 Cyphers must be put before the first Figure.

If the 8th Power of the Fraction, .05 be required, by multiplying the Logarithm by 8, there will be produced 69. 5917600, and since 70, which is seven times the Index of the Logarithm of Unity, cannot be taken from 69, unless we run into negative Numbers, the Index of the Logarithm of Unity must be supposed 100, and then the Index of the Logarithm of the Fraction will be 98. Now this Logarithm drawn into 8 gives 789. 5917600, and if 700, which is 7 times the Index of the Logarithm of Unity be taken from 789, there will remain 89. 5917600, the Logarithm of the 8th Power of the Fraction $\frac{1}{20}$, whose correspondent Number is, .0000000000 39062, for since the Index is 89, and the Difference thereof from 100 is 11; the first significant Figure of the Fraction shall be in the 11th Place from Unity; and so there must 10 Cyphers be placed before it.

If the Roots of the Powers of Fractions be desired, for Example, the Square Root of the Fraction E F, because the Root is a mean Proportional between the Fraction and Unity, you must bisect A E in C, and then C D will be the Square Root of the Fraction E F.

OA—OE

But AC = $\frac{1}{2}$ A E = $\frac{OA+OE}{2}$, and so OC the Loga-

OA+OE

rithm of the Root = OA—AC = $\frac{OA+OE}{2}$ And

if the Cube Root of the Fraction G H be sought,

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sought, this shall be the first of two mean Proportionals between Unity and G H ; and so if A.G be divided into three equal Parts, the first of which is A C ; then C D shall be the Root sought, and because $AC = \frac{1}{3} AG$
 $OA - OG$

$= \frac{2}{3} AG$, if this be taken from O.A, there

will remain $\frac{2}{3} OA + OG = OC = \text{Loga-}$

rithm of the Cube Root of the Fraction G H. So likewise the Biquadrat Root of the Fraction I K will be had, by dividing A I into four equal Parts, for the Root is the first of three mean Proportionals between Unity and the Fraction, and consequently if $AC = \frac{1}{4} AI$, then will C D, be the Biquadrat Root of the Fraction I K. But $\frac{1}{4} AI = \frac{1}{4} OA - OI$

and so

$OC = OA - AC = \frac{3}{4} OA + OI$

And universally, if the Root of any Power n of the Fraction L M be required, the Logarithm of the Root thereof will be $n OA - OA + OL$

, that is, if the Num-

ber $n - 1$ be prefix'd to the Index of the Logarithm, and the Logarithm thus augmented be divided by n , the Quotient will give the Logarithm of the Root sought. So if the Cube Root of the Fraction $\frac{1}{8}$ or .5 be sought, you must place $2 = n - 1$ (since the Cube Root

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Root is required) before the Logarithm thereof, and there will be had 29,6989700, a third Part of which is 9,8996566, which is equal to the Logarithm of the Cube Root of the Fraction $\frac{1}{3}$, and the Number ,7937 answering to this Logarithm, is the Root sought.



C H A P. IV.

Of the Golden Rule of Proportion by Logarithms.

THE Rule of Proportion shews how, by having three Numbers given, a fourth Proportional to them may be found, viz. if the second and third Terms be multiplied by one another, and the Product divided by the first Term, then will the Quotient be the fourth Proportional Term sought. But this fourth Term is much easier found by Logarithms; for if the Logarithm of the first Term be taken from the Sum of the Logarithms of the second and third Term, the Number remaining will be the Logarithm of the fourth sought.

Or this may be done something easier yet, if instead of the Logarithm of the first Term be taken it's Complement Arithmetical, or the Difference of that Logarithm, and the Number

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Number 10. 0000000, which is done by setting down the Difference between each Figure of the Logarithm, and the Figure 9; for then if that Arithmetical Complement be added to the Sum of the other two Logarithms, and if Unity, which is the first Figure to the Left-Hand, be taken from the Sum, the Remainder will be the Logarithm of the fourth Term sought; and so by this way Logarithms of the fourth Term is found by only one Addition of three Numbers. The Reason of this will be manifest from hence: Let there be three Numbers A, B, C, from which the first is to be taken from the Sum of the second and third. Now this may not only be done by the common Way, but likewise, if there be any other third Number E, taken, and from this there be taken A, there will remain $E - A$, and if the Numbers B, C, and $E - A$ be all added together, and from their Sum be taken E, there will remain $B + C - A$. So if the Number 15 be to be taken from 23, take the Complement of the Number 15 to 100, which is 85, and add this Number to 23, and the Sum will be 108, from which 100 being taken, there remains the Number 8.

Here follow some Trigonometrical Examples of the Rule of Proportion solv'd by Logarithms.

Let ABC be a Right-lined Triangle, wherein are given, the Angle A 36 Degrees 46', the Angle B 98 Degrees 32', and the Side BC 3478, the Side AC is required. Say (by Case 1. of Plain Trig.) as the Sine of the Angle

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Angle A is	Arith. Comp. S, A.	0. 2228938
to the Sine	Log. Sin. B.	9. 9951656
of the An-	Log. B C.	3. 5413296
gle B, so is	Log. A C.	1. 3. 7593896
BC to AC.		

And because

the Logarithm Sine of the Angle A is the first Term of the Analogy, I substitute it's Complement-Arithmetical for the same, and add the Logarithm of B C, the Logarithm of S, B, and the said Complement all three together, and reject Unity, which is in the first Place to the Left-Hand, and then the Logarithm of the Side A C will be given, and the Number answering thereto is 5706306 equal to the Side sought A C.

Let there be a spherical Triangle A B C in which are given all the Sides, viz. B C = 30 Degrees, A B = 24 Degrees 4', and A C = 42 Degrees 8', the Angle B is required. Let B A be produced to M, so that B M = B C, then will A M the Difference of the Sides B C, B A, be equal to 5 Degree 56'. Now (by Case 11. in Oblique-angle Spherical-Triangles) say, as the Rectangle under the Sines of the Legs, A B. B C. is to the Square of Radius, so is the Rectangle under the

$$\frac{A C + A M}{A C - A M} \text{ Sines of the Arcs } \frac{A C + A M}{A C - A M} \text{ to the}$$

$$\frac{AC + AM}{AC - AM} \text{ Square of the Sine of one half the Angle B}$$

But
$$\frac{AC + AM}{AC - AM} = 24 \text{ Degrees } 2', \text{ and } \frac{AC - AM}{AC + AM} = 18 \text{ Degrees } 6';$$
 and because the first Term

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of the Analogy is the Rectangle under the Sine of A B, B C, and second Term is the Square of Radius, the Sum of the Logarithm Sines of A B, B C, must be taken from double the Logarithm of Radius, and what remains must be added to the Sum of the Logarithm

$AC + AM$ $AC - AM$,
S, of $\frac{\quad}{2}$, & $\frac{\quad}{2}$ which is

the same as if the Logarithm Sines of each of the Arcs A B, B C, were subtracted from the Log.

Log. S, BC Comp. Arith. o. 3010299 of Radi-

Log. S, AB Comp. Arith. o. 3898364 us, or if

$AC + AM$

Log. S, $\frac{\quad}{2}$ 9.6098803 the Com-

$AC - AM$

Log. S, $\frac{\quad}{2}$ 9.4923083 Arith -

2. Log. S, Angle B. 19.7930549 metical

of these Sines be taken, & the Com-

plements and the said Sines be all added to-

gether ; then shall the Sum be the Logarithm

of the Square of the Sine of half the Angle

B. And so the half of the Logarithm

9.8965274 is the Logarithm Sine of half the

Angle B = 51 Degrées 59'. 56", and the

Double of this Angle shall be 103 Degrées

59', 52" = B, which was sought.



C H A P. V.

Of the continual Increments of proportional Quantities, and how to find by Logarithms any Term in a Series of Proportionals, either increasing or decreasing.
Fig. 3.



IF any where in the Axis of the Logarithmical Curve, there be taken any Number of equal Parts $S V, V Y, Y Q, \&c.$ and at the Points $S, V, Y, Q, \&c.$ be raised the Perpendiculars $ST, VX, YZ, Q\pi, \&c.$ then from the Nature of the Curve shall all these Perpendiculars be continually proportional; and therefore also the continual Increments $Xx, Zz, \pi\pi,$ shall be proportional to their Wholes: For since $ST: VX:: VX: YZ:: YZ: Q\pi$, it shall be (by Division of Proportion) $ST: Xx:: VX: Zz:: YZ: \pi\pi$, and (by Composition of Proportion) $VX: Xx:: YZ: Zz:: Q\pi: \pi\pi$. Hence, if Xx be any part of any Right Line ST , then will Zz be the same Part of the Right Line

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Line VX , and also n the same Part of the Right-Line YZ . For Example; if X be the $\frac{1}{10}$ Part of ST , then will $Zz = \frac{1}{10} VX$, and $n = \frac{1}{10} YZ$; or which comes to the same, we shall have $VX = ST + \frac{1}{10} ST$, $YZ = VX + \frac{1}{10} VX$. Also $Qn = YZ + \frac{1}{10} YZ$.

Now make, as ST is to VX , so is Unity AB to NR ; then shall $AN = SV$; and so each of the Right Lines SV, VY, YQ , &c. shall be equal to the Logarithm of NR , and AV , the Logarithm of the Term VX shall be equal to $AS + AN = \text{Logarithm of } ST + \text{Logarithm of } NR$. Also AY , the Logarithm of the Term YZ , shall be equal to $AS + 2AN = \text{Logarithm } ST + 2 \text{ Logarithm } NR$, and AQ , the Logarithm of the Term Qn shall be equal to $AS + 3AN = \text{Logarithm } ST + 3 \text{ Logarithm } NR$. And universally, if the Logarithm of the Number NR be multiplied by a Number, expressing the Distance of any Term from the first, and the Product be added to the Logarithm of the first Term, then will the Logarithm of that Term be had: But if a Series of Proportionals be decreasing, that is, if the Terms diminish in a continual Ratio, and Qn be the first Term; then the Logarithm of any other will be had, in multiplying the Logarithm of the Number NR , by a Number that expresses the Distance of it's Term from the first, and subtracting the Product from the Logarithm of the first. And if the said Product be greater than the Logarithm of the first Term, beginning from Unity, then the Logarithms must

R begin

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begin from a Unit in some Place of Decimal Fractions, as from O P, and then the Logarithm of the Number Q n will be O Q.

Now let L M represent any Money, or Sum of Money put out to Interest, so that the Interest thereof be accounted but at the End of every Year, and let K k be the Gain or Interest thereof at the End of the first Year, then will I K be the Sum of the Interest and Principal. And again, I K becoming the Principal at the End of the first Year, H h which is proportional to I K, or in a constant Ratio, will be the Gain at the end of the second Year; and so H G, at the End of the second Year, will become the Principal; and at the end of the third Year F f, proportional to G H, will be the Gain. Now let us suppose the Principal be augmented every Year $\frac{1}{10}$ Part thereof, so that $I K = L M + \frac{1}{10} L M$, $G H = I K + \frac{1}{10} I K$, $E F = G H + \frac{1}{10} G H$, and so on. And accordingly the Terms L M, I K, G H, E F, &c. continual Proportionals, it is required to find the Amount of the Money at the End of any Number of Years.

Let L M be a Farthing. Because L M is to I K, as 1 to $1 + \frac{1}{10}$, or as 1 to 1.05, as A B is to N R, then will $N R = 1.05$, whose Logarithm A N. is 0.0211893, or more accurately 0.0211892991, it is required to find the Amount of a Farthing put out at compound Interest, at the End of 600 Years, multiply A N by 600, and the Product will be 12.7135794, and to this Product add the Logarithm of the Fraction $\frac{1}{10}$, viz. 97.0177288 (for

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(for a Farthing is $\frac{1}{4}$ Part of a Pound) and the Sum 109. 7313082 shall be the Logarithm of the Number sought; and since the Index 109 exceeds the Index of Unity by 9, there shall be nine Places of Figures above Unity in the correspondent Number, and that Number being sought in the Tables, will be found greater than 5386500000, and less than 5386600000. And therefore a Farthing put out at Interest upon Interest, at 5 *per Cent.* per *Ann.*, at the End of 600 Years will amount to above 5386500000 Pounds; which Sum could hardly be made up by all the Gold and Silver that has been dug out of the Bowels of the Earth, from the beginning of the World to this Time.

Let Qn expound any Sum of Money due to some Person at the End of a full Year, but without Interest. Now it is certain, that if the Debtor should pay down present the whole Sum of Money, he would lose the yearly Usury or Interest that his Money would gain him; and so a lesser Sum, being put out to Interest, will at the End of one Year, together with the Interest thereof, be equal to the Sum of Money Qn . Now this present Sum of Money, which together with the Interest thereof is equal to the Sum of Money Qn , is called the present Worth of the Money Qn . Let AN be the Logarithm of the Ratio, which the Principal has to the Sum of the Principal and Interest, that is, if the Principal be twenty times the yearly Interest, let AN be the Logarithm of the Number $1 + \frac{1}{20}$ or 1.05, and take QI equal to AN ; then will AY

be the Logarithm of the present Worth of Money Qn . For it is manifest, that the Money YZ , put out to Interest, will at the End of one Year amount to the Money Qn , and so to have the Logarithm of the present Worth thereof, or YZ , the Logarithm AN , must be taken from the Logarithm AQ , and there will remain the Logarithm AV of the present Worth, or YZ . But if the Sum nQ be not due till the End of two Years, then the Logarithm $2AN$ must be subtracted from the Logarithm AQ , and there will remain AV , the Logarithm of the present Worth, or of the Sum that must be paid down present for the Money Qn due at the End of two Years. For it is manifest, that the Money VX being put out to Interest, will at the End of two Years amount to the Sum of Money Qn . By the same Reason, if the Sum Qn , be not due until the End of three Years, the Logarithm $3AN$ must be subtracted from the Logarithm of Qn , and the Remainder AS , shall be the Logarithm of the Number ST , or ST shall be the present Worth of the Sum Qn due at the three Years end. And universally, if the Logarithm AN be multiplied by the Number of Years, at the End of which the Sum Qn is due, and the Number produced be taken from the Logarithm AQ , then will the Logarithm of the present Worth of the Sum Qn be had. And from hence it is manifest, if 5386500000 Pounds be due to some Society at the End of 600 Years, then would the present Worth of that vast Sum of Money be scarcely a Farthing. If

If the proportional Right Lines $H G$, $E F$, $A B$, $C D$, *Fig. 4.* are Ordinates to the Axis of the Logarithmical-Curve, and if their Ends $F H$, $D B$, be joined by Right Lines, which produced meet the Axis in the Points P and K , then the Right Lines $G P$, $A K$, will be always equal. For since $G H : E F :: A B : C D$. it will be as $G H : F s :: A B : D R$. But because of the equiangular Triangles $P G H$, $H s F$, as also $K A B$, $B R D$, we have $P G : H s :: (G H : F s :: A B : D R :) K A : B R$. And since the Consequents $H s$, $B R$, are equal, the Antecedents $P G$, $K A$, shall be also equal. *W. W. D.*

If the Right Lines $C D$, $E F$, equally accede to $A B$, $G H$, so that the Point D at last may coincide with B , and the Point F with H , then the Right Lines $D B K$, $F H P$, which did cut the Curve before, will be changed into the Tangents $B T$, $H V$. And the Right Lines $A T$, $G V$, will be always equal to each other, that is, the Portion of the Axis $A T$, or $G V$, intercepted between the Ordinate and the Tangent, which is called the Subtangent, will every where be a constant and given Length. And this is one of the chief Properties of the Logarithmical Curve; for the different Species or Forms of those Curves are determined by the Subtangents.

The Logarithms or the Distances from Unity of the same Number, in two Logarithmical-Curves of different Species, will be proportional to the Subtangents of their Curves. For let $H B D$, $S N Y$, *Fig. 4, 5.* be
Curves,

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Curves, whose Subtangents are AT , MX , and let $AB = MN = \text{Unity}$; also $DC = QY$, then shall AC the Logarithm of the Number CD , in the Logarithmical-Curve HD be to MQ , the Logarithm of the Number QY , (or of the said CD), in the Curve SY , as the Subtangent AT is to the Subtangent MX . For let there be supposed an infinite Number of mean proportional Terms between AB , CD , or NM , QY , in the Ratio of AB to ab , or MN to mn ; and since $AB = MN$, then will $ab = mn$, as also $bc = no$. And because the Number of proportional Terms in each Figure are equal, they do divide the Lines AC , MQ , into equal Numbers of Parts, the first of which Aa , Mm , and so the said Parts shall be proportional to their Wholes, that is, it will be as $Aa : Mm :: AC : MQ$. And because the Triangles TAB , Bcb , are similar, (for the Part of the Curve Bb nearly coincides with the Portion of the Tangent,) as also the Triangles XMN , $No n$, are similar, we have Aa , or $Bc : bc :: TA : AB$.

Also as no , or $bc : No :: MN$, or $AB : MX$.

Where (by Equality of Proportion) it will be $Bc : No :: TA : MX :: Aa : Mm :: AC : MQ$; *which was to be demonstrated*. If AT be call'd a , since $AB : AT :: bc : Bc$, then will $Bc = \frac{a \times bc}{AB}$.

Hence, if the Logarithm of a Number extremely near Unity, or but a small matter exceeding it, be given, then will the Subtangent

gent of the Logarithmical-Curve be had. For the Excess bc is to the Logarithm Bc , as Unity AB is to the Subtangent AT . Or even if there are any two Numbers nearly equal, their Difference shall be to the Difference of their Logarithms, as one of the Numbers is to the Subtangent. For Example ; if the Increment bc be ,00000 00000 00001 02255 31945 60259, and Bc or Aa the Logarithm of the Number ab be ,00000 00000 00000 44408 92098 50062. Now if a fourth Proportional be found to the said two Numbers and Unity, *viz.* 43429 44819 03251, this Number will give the Length of the Subtangent AT , which is the Subtangent of the Curve, expressing *Brigg's* Logarithms.

If a Sum of Money be put out to Interest on this Condition, That a proportional Part of the yearly Rate of Interest thereof, be accounted every Moment of Time, *viz.* so, that at the End of the first Moment of Time, or indefinitely small Particle of a Year, the Interest gotten thereby, be proportional to that Time ; which being added to the Principal, again begets Interest at the End of the second Moment of Time, and then the Principal, and this Interest become a Principal, and so on. It is requir'd to find the Amount of that Sum at the Year's End. Let a be nearly the Interest of Unity, or of one Pound. Then if one whole Year, or 1 gives the Interest a , the indefinitely small Particle of a Year Mm , will give the Interest $Mm \times a$, proportional to Mm ; and accordingly, if Unity be expounded by MN , the first Increment

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crement thereof shall be $no = Mm \times a$. This being granted, let a Logarithmical Curve be suppos'd to be described through the Point Nn , whose Axis is OMQ . Then in this Curve, if the Portion of the Axis MQ expresses the Time, the ordinate Qy will represent the Money proportionally increasing every Moment, to that Time. For if there be taken ml , &c. $= Mm$, the Ordinates lp , &c. shall be in a Series of continual Proportionals in the Ratio of MN to mn , that is, they increase in the same Ratio, as the Money doth.

Again, let the Right-Line NX touch the Logarithmical-Curve in N , and the Subtangent thereof MX , shall be constant and invariable, and the small Triangle $No n$, shall be similar to the Triangle XMN . But it has been prov'd, that the Increment $no = Mm \times a = No \times a$; and so $no : No :: No \times a : No :: a : 1$. But as no is to No , so shall NM be to MX . Wherefore it shall be as a is to 1, so is NM , or 1, to $MX = \frac{1}{a} = \text{Subtangent}$.

Now if the yearly Rate of Interest be $\frac{5}{100}$ part of the Principal, or if $a = \frac{5}{100} = .05$, then will $MX = \frac{1}{.05} = 20$.

Because in different Forms of Logarithms, the Logarithms of the same Number, are proportional to the Subtangents of their Curves: If MQ expresses the Time of a whole Year, or Unity, then shall QY be the Amount of the Money at the Year's End. And to find QY , say, as MX , or 20 is to 0.4342944, (which Number expounds the Subtan-

Subtangent of the Logarithmical-Curve, expressing *Brigg's* Logarithms) so is one Year or Unity to a *Briggian* Logarithm, answering to the Number QY. This Logarithm will be found 0.0217147, and the Number answering to the same is $1.05127 = QY$, whose Increment above Unity, or the Principal, exceeds the yearly Interest .05 but a small matter. And so if the yearly Interest of 100 Pounds be 5 Pounds, the proportional yearly Interest, which is added to the Principal 100, at the end of each Particulate of the Year, will amount only at the Year's End to 5 Pounds, 2 Shillings and $6\frac{1}{2}$ Pence.

And if such a Rate of Interest be requir'd, that every Moment a Part of it continually proportional to the increasing Principal, be added to the Principal, so that at the Year's End, an Increment be produc'd, that shall be any given Part of the Principal, for Example, the $\frac{1}{20}$ Part, say, as the Logarithm of the Number 1.05 is to 1; that is, as 0.0211893 is to 1, so is the Subtangent 0.4342944 to $\frac{1}{20} = 20.49$, and then will $a = \frac{1}{20.49} = .0488$. For if such a Part of the Rate of Interest .0488 be supposed as answers to a Moment, that is, having the same Ratio to .0488, as a Moment has to a Year, and it be made, as Unity is to that Part of the Rate of Interest, so is the Principal to the Momentaneous-Increment thereof; then will the Money continually increasing in that manner, be augmented at the Year's-End the $\frac{1}{20}$ Part thereof.



C H A P. VI

Of the Method by which Mr. Briggs computed his Logarithms, and the Demonstration thereof.



LTHO' Mr. Briggs has nowhere describ'd the Logarithmical-Curve, yet it is very certain, that from the Use and Contemplation thereof, the Manner and Reason of his Calculations will appear. In any Logarithmical-Curve HBD , let there be three Ordinates AB, ab, qs , nearly equal to one another; that is, let their Differences have a very small Ratio to the said Ordinates; and then the Differences of their Logarithms will be proportional to the Differences of the Ordinates. For since the Ordinates are nearly equal to one another, they will be very nigh to each other, and so the Part of the Curve Bs , intercepted by them, will almost coincide with a strait Line; for it is certain, that the Ordinates may be so near to each other, that the Difference between the Part of the Curve, and the Right-Line subtending it, may have to that Subtense, a Ratio less than any given Ratio. Therefore
the

the Triangles Bcb , Brs , may be taken for Right-lin'd, and will be Equiangular. Wherefore, as $sr : bc :: Br : Bc :: Aq : Aa$; that is, the Excesses of the Ordinates, or Lines above the least, shall be proportional to the Differences of their Logarithms. And from hence appears the Reason of the Correction of Numbers and Logarithms, by Differences and proportional Parts. But if AB be Unity, the Logarithms of Numbers shall be proportional to the Differences of the Numbers.

If a mean Proportional be found between 1 and 10, or which is the same thing, if the Square-Root of 10 be extracted, this Root or Number will be in the middle Place between Unity and the Number 10, and the Logarithm thereof shall be $\frac{1}{2}$ of the Logarithm of 10, and so will be given. If again, between the Number before found, and Unity, there be found a mean Proportional, which may be done in extracting the Square-Root of the said Number, this Number or Root will be twice nearer to Unity than the former, and its Logarithm will be one half of the Logarithm of that, or one fourth of the Logarithm of 10. And if in this Manner, the Square-Root be continually extracted, and the Logarithms bisected, you will at last get a Number, whose Distance from Uni-

1

ty, shall be less than the $\frac{1}{100000000000000}$ part of the Logarithm of 10. And after Mr. Briggs had made 54 Extractions of the Square-Root, he found the Number 1.00000

00000 00000 12781 91493 20032 3442, and its Logarithm was 0.00000 00000 00000,05551 115123125782702. Suppose this Logarithm to be equal to Aq or Br , and let qs be the Number found, by extracting the Square-Root; then will the Excess of this Number above Unity, viz. $rs = 0,00000 00000 00000$. 12781 91493 20032 34.

Now by means of these Numbers, the Logarithms of all other Numbers, may be found in the following Manner: Between the given Number (whose Logarithm is to be found) and Unity, find so many mean Proportionals, (as above,) till at last a Number be gotten so little exceeding Unity, that there be 15 Cyphers next after it, and a like Number of significant Figures after those. Let this Number be ab , and let the significant Figures, with the Cyphers prefixed before them, denote the Difference bc . Then say, As the Difference rs , is to the Difference bc , so is Br a given Logarithm, to Bc , or Aa , the Logarithm of the Number ab ; which therefore is given. And if this Logarithm be continually doubled, the same Number of Times, as there were Extractions of the Square-Root, you will at last have the Logarithm of the Number sought. Also by this Way, may the Subtangent of the Logarithmical-Curve be found, viz. in saying, As $rs : Br :: AB$, or Unity : AT , the Subtangent, which therefore will be found to be 0.43429 44819 03251; by which may be found the Logarithms of other Numbers; to wit, if any Number NM be given afterwards,

wards, as also its Logarithm, and the Logarithm of another Number sufficiently near to NM be sought, say, As NM is to the Subtangent XM , so is no the Difference of the Numbers to N the Difference of the Logarithms. Now, if NM be Unity $= AB$, the Logarithms will be had by multiplying the small Differences bc by the constant Subtangent AT .

By this Way may be found the Logarithms of 2, 3, and 7, and by these the Logarithms of 4, 8, 16, 32, 64, &c. 9, 27, 81, 243, &c. as also 7, 49, 343, &c. And if from the Logarithm of 10, be taken the Logarithm of 2, there will remain the Logarithm of 5, so there will be given the Logarithms of 25, 125, 625, &c.

The Logarithms of Numbers compounded of the aforesaid Numbers, viz. 6, 12, 14, 15, 18, 20, 21, 24, 28, &c. are easily had by adding together the Logarithms of the component Numbers.

But since it was very tedious and laborious, to find the Logarithms of the Prime-Numbers, and not easy to compute Logarithms by Interpolation, by first, second and third, &c. Differences, therefore the great Men, Sir *Isaac Newton*, *Mercator*, *Gregory*, *Wallis*, and lastly, Dr. *Halley*, have published infinite converging Series, by which the Logarithms of Numbers to any Number of Places, may be had more expeditiously and truer: Concerning which Series, Dr. *Halley* has written a learned Tract, in the *Philosophical-Transactions*, wherein he has demonstrated those

those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by means of which may be found easily and expeditiously, the Logarithms of large Numbers.

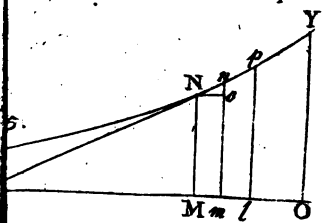
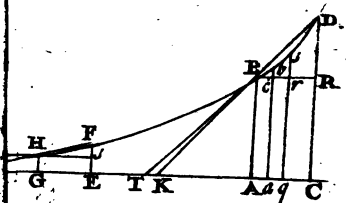
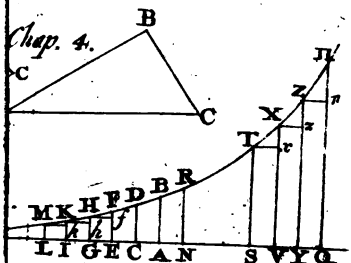
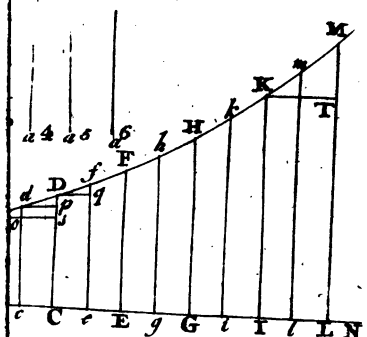
Let z be an odd Number, whose Logarithm is sought; then shall the Numbers $z-1$ and $z+1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms will be had, which let be called y : Therefore, also the Logarithm of a Number, which is a Geometrical-Mean between $z-1$ and $z+1$ will be given, *viz.* equal to the half Sum of the Logarithms. Now the Series

$$y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9} \text{ &c.}$$

shall be equal to the Logarithm of the Ratio, which the Geometrical-Mean between the Numbers $z-1$ and $z+1$, has to the Arithmetical-Mean, *viz.* to the Number z .

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by Briggs. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2, with the Index 4 prefix'd to it; and the

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the Difference of the Logarithms of 20000 and 20002, is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, viz. 0.00004 34272 7687. And if this Difference be divided by 42, or 80004, the

Quotient $\frac{2}{42}$ shall be 0.00000 00005 42813

And if the Logarithm 4.30105 17093 02416 of the Geometrical-Mean be added to the

Quotient, the Sum will be the Logarithm of 20001. Wherefore it is manifest, that to have the Logarithm to 14 Places of Figures, there is no Necessity of continuing out the Quotient beyond six Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq's* Tables, the two first Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000, are to be found by this Way, the Labour of doing them, will mostly consist in setting down the Numbers. *Note*, This Series is easily deduced from that found out by *Dr. Halley*; and those who have a mind to be inform'd more in this Matter, let them consult his above-nam'd Treatise.

F I N I S.

An Explanation of the usual Algebraic Signs or Characters.

+	<i>More, or to be added to</i>
-	<i>Less, or to be subtracted from</i>
=	<i>Equal to</i>
x	<i>Multiplied by</i>
$\frac{a}{b}$	<i>a divided by b</i>
a : b :: c : d	<i>As a is to b, so is c to d</i>
° ' " "	<i>Degrees, Minutes, Seconds, Thirds</i>
52°. 44'. 3"	<i>52 Seconds, 44 Thirds, 3 Fourths</i>
R.	<i>Radius, or Sine of 90 Degrees</i>
S.	<i>Sine</i>
Cos.	<i>Cosine, or Sine of the Complement</i>
T.	<i>Tangent</i>
Cot.	<i>Cotangent, or Tangent of the Complement</i>
V. S.	<i>Versed-Sine</i>
□ or q	<i>Square</i>
√ or √²	<i>Square Root</i>
A³ or B⁴	<i>Cube of A. or 4th Power of B.</i>
√³ or √⁴	<i>Cube Root, or Biquadrate Root (Euclid</i>
by 4. 6	<i>by the 4th Proposition of the 6th Book of</i>
W. W. D.	<i>which was to be demonstrated or done</i>
N°. S°.	<i>North or South</i>
♈	<i>Aries or Libra</i>
♋	<i>Cancer or Capricorn</i>
☉ ☾	<i>Sun or Moon.</i>

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